Objectives

Dynamic Programming

- Shortest paths
- Distance Vector Protocol

Network flow

- Maximum flow
- Minimum cuts

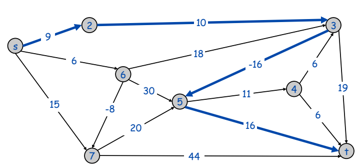
SHORTEST PATHS

Shortest Paths

Given a directed graph G = (V, E), with edge weights c_{vw}, find shortest path from node *s* to node *t*

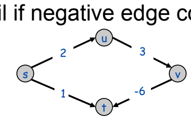
allow negative weights

Allows modeling other phenomena



Shortest Paths: Failed Attempts

Dijkstra. Can fail if negative edge costs



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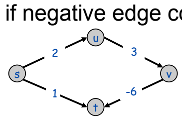
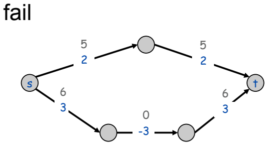
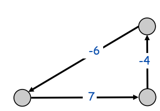


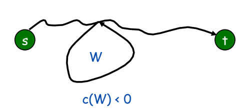
Re-weighting. Adding a constant to every edge weight can fail

Why?



Shortest Paths: Negative Cost Cycles





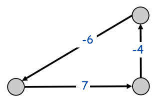
If some path from *s* to *t* contains a negative cost cycle, there does **not** exist a shortest *s*-*t* path

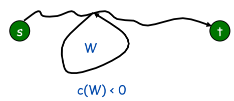
Why?

Otherwise, there exists one that is *simple* (i.e., does not repeat nodes)

- What does this mean about number of edges in path?

Shortest Paths: Negative Cost Cycles





If some path from *s* to *t* contains a negative cost cycle, there does **not** exist a shortest *s-t* path

Otherwise, there exists one that is *simple* (i.e., does not repeat nodes)

- Path has *at most n-1* edges, where *n* is # of nodes in graph

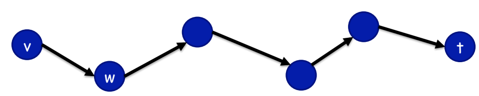
Towards a Recurrence

OPT(*i*,*v*): minimum cost of a *v-t* path *P* using *at most i* edges

- This formulation eases later discussion

Original problem is OPT(n-1, s)

Break down into subproblems based on *i* and *v*



Path P

Shortest Paths: Dynamic Programming

Def. OPT(i, v) = minimum cost of a *v-t* path *P* using at most *i* edges

- Case 1: *P* uses at most *i-1* edges
  - OPT(i, v) = OPT(i-1, v)
- Case 2: *P* uses exactly *i* edges
  - if (*v, w*) is first edge, then OPT uses (*v, w*), and then selects best *w-t* path using at most *i-1* edges
  - Cost: cost of chosen edge

$$OPT(i,v) = \begin{cases} 0 & \text{if } i=0 \\ \min\left\{ OPT(i-1, v),\ \min_{(v,w)\in E} \{ OPT(i-1, w) + c_{vw} \} \right\} & \text{otherwise} \end{cases}$$

Shortest Paths: Implementation

```
Shortest-Path(G, t)
  n = number of nodes in G
  foreach node v ∈ V
    M[0, v] = ∞     # infinite cost to reach all nodes
  M[0, t] = 0      # no cost to reach destination from dest

  for i = 1 to n-1
    foreach node v ∈ V
      M[i, v] = M[i-1, v]  # at most cost of 1 less
      foreach edge (v, w) ∈ E
        M[i, v] = min(M[i, v], M[i-1, w] + c_vw )
```

Analysis?

Cost of chosen edge

Shortest path is M[n-1, s]

Starting node

Shortest Paths: Implementation

```
Shortest-Path(G, t)
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      foreach edge (v, w) ∈ E
        M[i, v] = min(M[i, v], M[i-1, w] + c_vw )
```
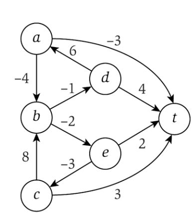

$O(n^3)$

Shortest path is M[n-1, s]

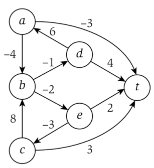
Example



	0	1	2	3	4	5
t	0	0	0	0	0	0
a	∞					
b	∞					
c	∞					
d	∞					
e	∞					

What edges do we need to look at for each node?

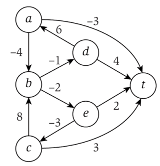
Example



Edges
- b , t
- d, e
- b, t
- a, t
- c, t

	0	1	2	3	4	5
t	0	0	0	0	0	0
a	∞					
b	∞					
c	∞					
d	∞					
e	∞					

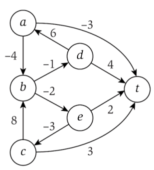
Example



Edges
- b , t
- d, e
- b, t
- a, t
- c, t

	0	1	2	3	4	5
t	0	0	0	0	0	0
a	∞	-3				
b	∞	∞				
c	∞	3				
d	∞	4				
e	∞	2				

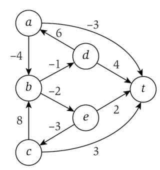
Example



Edges
- b , t
- d, e
- b, t
- a, t
- c, t

	0	1	2	3	4	5
t	0	0	0	0	0	0
a	∞	-3	-3			
b	∞	∞	0			
c	∞	3	3			
d	∞	4	3			
e	∞	2	0			

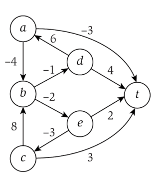
Example



Edges
- b , t
- d, e
- b, t
- a, t
- c, t

	0	1	2	3	4	5
t	0	0	0	0	0	0
a	∞	-3	-3	-4		
b	∞	∞	0	-2		
c	∞	3	3	3		
d	∞	4	3	3		
e	∞	2	0	0		

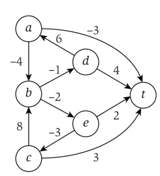
Example



Edges
- b , t
- d, e
- b, t
- a, t
- c, t

	0	1	2	3	4	5
t	0	0	0	0	0	0
a	∞	-3	-3	-4	-6	
b	∞	∞	0	-2	-2	
c	∞	3	3	3	3	
d	∞	4	3	3	2	
e	∞	2	0	0	0	

Example



Edges
- b , t
- d, e
- b, t
- a, t
- c, t

	0	1	2	3	4	5
t	0	0	0	0	0	0
a	∞	-3	-3	-4	-6	-6
b	∞	∞	0	-2	-2	-2
c	∞	3	3	3	3	3
d	∞	4	3	3	2	0
e	∞	2	0	0	0	0

Based on Example Experience

What could we do to improve the algorithm's runtime/space requirements?

Shortest Paths: Practical Improvements

Practical improvements

- Maintain only one array M[v] = shortest *v*-t path that we have found so far
- No need to check edges of the form (*v*, *w*) ***unless*** M[w] changed in previous iteration

Theorem. Throughout algorithm, M[v] is length of some *v-t* path, and after *i* rounds of updates, the value M[v] is no larger than the length of shortest *v-t* path using ≤ *i* edges.

Overall impact

- Memory: O(m + n)
- Running time: O(mn) worst case but substantially faster in practice

Bellman-Ford: Efficient Implementation

```
Push-Based-Shortest-Path(G, s, t)
   foreach node v ∈ V
      M[v] = ∞
      successor[v] = ϕ

   M[t] = 0
   for i = 1 to n-1
      foreach node w ∈ V
         if M[w] has been updated in previous iteration
            foreach node v such that (v, w) ∈ E
               if M[v] > M[w] + c_vw
                  M[v] = M[w] + c_vw
                  successor[v] = w

      If no M[w] value changed in iteration i, stop.
```


DISTANCE VECTOR PROTOCOL

Problem Context

Application of shortest-path problem: *routers in communication network find most efficient path to destination*

Model of communication network

- Nodes ≈ routers
- Edge ≈ direct communication link
- Cost of edge ≈ delay on link ← *Naturally nonnegative*

Possible solution: Dijkstra's algorithm

Distance Vector Protocol

Model of communication network

- Nodes ≈ routers; Edge ≈ direct communication link
- Cost of edge ≈ delay on link ← *Naturally nonnegative but Bellman-Ford used anyway!*

Dijkstra's algorithm. Requires *global* information of network

Bellman-Ford. Uses only local knowledge of neighboring nodes

- **Distribute** algorithm: each node *v* maintains its value M[v]
  - Updates its value after getting neighbor's values:
    - $\min_{w \in V} (c_{vw} + M[w])$

Distance Vector Protocol

Each router maintains a vector of shortest path lengths to every other node (distances) and the first hop on each path (directions)

Algorithm: each router performs *n* separate computations, one for each potential destination node

Synchronization. We don't expect routers to run in lockstep. The order in which each `foreach` loop executes in not important. Moreover, algorithm still converges even if updates are asynchronous.

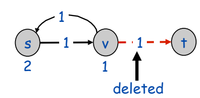
"Routing by rumor."

Used in many routers, e.g. RIP, Xerox XNS RIP, Novell's IPX RIP, …

Issues with Distance Vector Protocol

Original algorithm developed for one central machine; costs known in advance, didn't change

Edge costs may change during algorithm (or fail completely)



"counting to infinity"

Path Vector Protocols

Link state routing

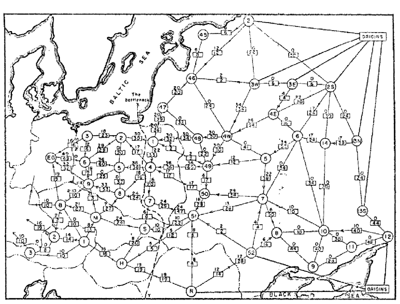
- Each router stores the *entire path*
  - Not just the distance and the first hop
- Based on Dijkstra's algorithm
- Avoids "counting-to-infinity" problem and related difficulties
- Requires significantly more storage

Ex. Border Gateway Protocol (BGP), Open Shortest Path First (OSPF)

NETWORK FLOW

Soviet Rail Network, 1955

44 vertices
105 edges



Reference: *On the history of the transportation and maximum flow problems.* Alexander Schrijver in Math Programming, 91: 3, 2002.

Maximum Flow and Minimum Cut

Two very rich algorithmic problems

Cornerstone problems in combinatorial optimization

Beautiful mathematical duality

Nontrivial applications / reductions

- Data mining
- Open-pit mining
- Project selection
- Airline scheduling
- Bipartite matching
- Baseball elimination
- Image segmentation
- Network connectivity
- Network reliability
- Distributed computing
- Egalitarian stable matching
- Security of statistical data
- Network intrusion detection
- Multi-camera scene reconstruction
- Many many more . . .

Flow Network

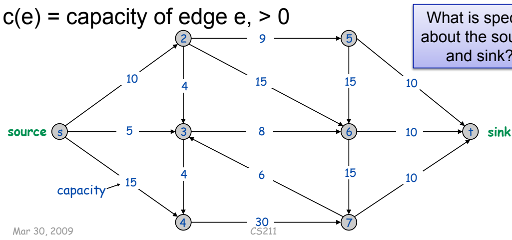
Abstraction for material *flowing* through the edges

G = (V, E) = directed graph, no parallel edges

Two distinguished nodes: *s* = source, *t* = sink

c(e) = capacity of edge e, > 0

What is special about the source and sink?



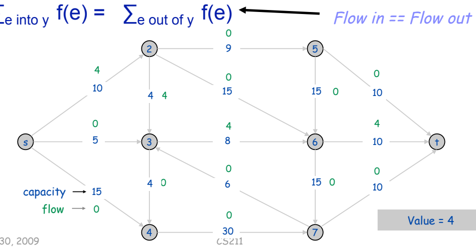
Flows

An **s-t flow** is a function that satisfies

- *Capacity* condition: For each $e \in E$: $0 \le f(e) \le c(e)$
- *Conservation* condition: For each $v \in V - \{s, t\}$:
  $\sum_{e \text{ into } y} f(e) = \sum_{e \text{ out of } y} f(e)$

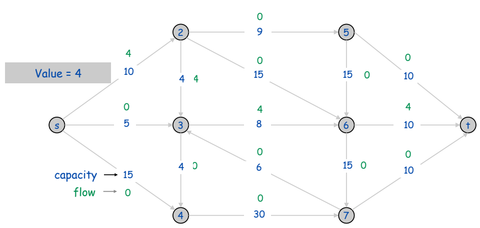
Flow can't exceed capacity

Flow in == Flow out



Flows

The value of a flow f is $v(f) = \sum_{e \text{ out of } s} f(e)$



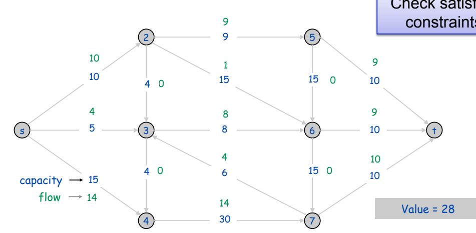
Maximum Flow Problem

Make network most efficient

- Use most of available capacity

Goal: Find *s-t* flow of maximum value

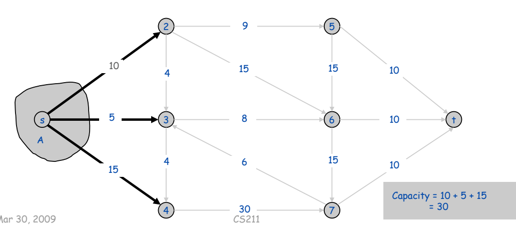
Check satisfies constraints



Cuts

An **s-t cut** is a partition (A, B) of V with $s \in A$ and $t \in B$

The **capacity** of a cut (A, B) is $cap(A, B) = \sum_{e \text{ out of } A} c(e)$



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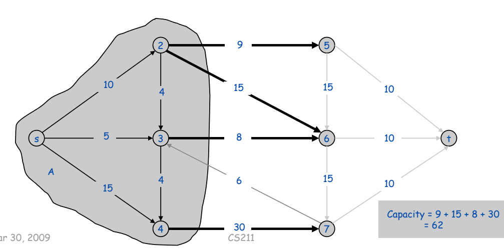
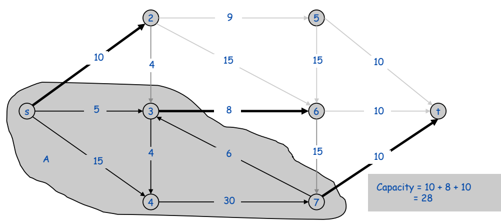


Minimum Cut Problem

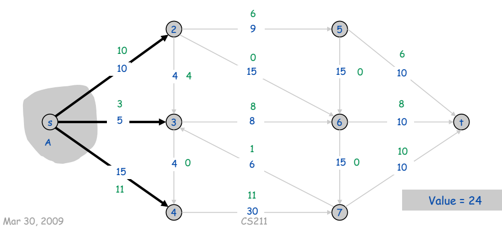
Find an *s-t* cut of *minimum* capacity

- Puts *upperbound* on maximum flow



Flow Value Lemma

Let *f* be any flow, and let (A, B) be any *s-t* cut. Then, the ***net flow*** sent across the cut is equal to the amount leaving *s*. $\sum_{e \text{ out of } A} f(e) - \sum_{e \text{ in to } A} f(e) = v(f)$



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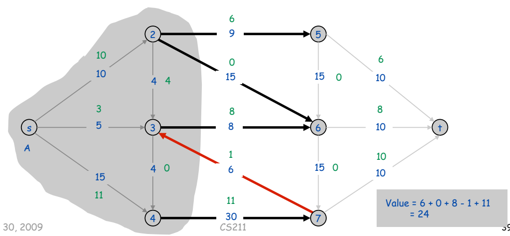

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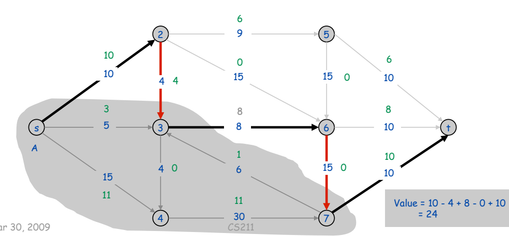

Flow Value Lemma

Let *f* be any flow, and let (A, B) be any *s-t* cut.

Then $\sum_{e \text{ out of } A} f(e) - \sum_{e \text{ in to } A} f(e) = v(f).$

Pf.

$$v(f) = \sum_{e \text{ out of } s} f(e)$$

by flow conservation, all terms except v = s are 0

$$= \sum_{v \in A} \left(\sum_{e \text{ out of } v} f(e) - \sum_{e \text{ in to } v} f(e) \right)$$

$$= \sum_{e \text{ out of } A} f(e) - \sum_{e \text{ in to } A} f(e).$$

This Week

Chapters 7 & 8

Wednesday: Course evaluations

- Favorite and least favorite topics
- Research papers
  - Do you think that component should continue?
    - Was it worth it?

Friday: Problem set 6 due

Saturday: Take-home final available

- Due end of exam period: Friday at 5 p.m.