Objectives

Reviewing Presentations

Stable Matching Problem

Road to a Dissertation and Beyond…

Course work beyond master's degree

Research proposal

- ~2 papers accepted or ready for submission

Dissertation

- Culmination of 4 papers
- Tell a story

After the dissertation

- A few more publications directly from the dissertation
- Expand

ALGORITHMS

Matching Residents to Hospitals

Goal: Given a set of preferences among hospitals and medical school students, design a self-reinforcing admissions process.

Unstable pair: applicant *x* and hospital *y* are unstable if:

- *x* prefers *y* to its assigned hospital
- *y* prefers *x* to one of its admitted students

Stable assignment: Assignment with no unstable pairs

- Natural and desirable condition
- Individual self-interest will prevent any applicant/hospital deal from being made

What details make this problem tricky?

Stable Matching Problem

Simplified version of resident-matching problem

Goal: Given *n* men and *n* women, find a "suitable" matching

- Participants rate members of opposite sex
- Each man lists women in order of preference from best to worst
- Each woman lists men in order of preference from best to worst

favorite ↓ least favorite ↓

	1st	2nd	3rd
Xavier	Amy	Bertha	Clare
Yancey	Bertha	Amy	Clare
Zeus	Amy	Bertha	Clare

Men's Preference Profile

favorite ↓ least favorite ↓

	1st	2nd	3rd
Amy	Yancey	Xavier	Zeus
Bertha	Xavier	Yancey	Zeus
Clare	Xavier	Yancey	Zeus

Women's Preference Profile

Stable Matching Problem

Perfect matching: everyone is matched monogamously

- Each man is paired with exactly one woman
- Each woman is paired with exactly one man

Stability: no incentive for some pair of participants to undermine assignment by joint action

- In matching M, an *unmatched* pair m-w is unstable if man *m* and woman *w* prefer each other to current partners
- Unstable pair m-w could each improve by eloping

Stable matching: perfect matching with no unstable pairs

Stable matching problem. Given the preference lists of *n* men and *n* women, find a stable matching if one exists.

Any Questions?

What are you wondering about this problem at this point?

- Is it possible to match everyone?
- Can we be fair in the matching? (preferences)
- Will the matching always be the same?

Propose-And-Reject Algorithm

[Gale-Shapley 1962]

Intuitive method that guarantees to find a stable matching

```
Initialize each person to be free.
while (some man is free and hasn't proposed to every woman) {
    Choose such a man m
    w = 1st woman on m's list to whom m has not yet proposed
    if (w is free)
        assign m and w to be engaged
    else if (w prefers m to her fiancé m')
        assign m and w to be engaged, and m' to be free
    else
        w rejects m
}
```


Observations about the Algorithm

What can we say about any woman's partner over the execution of the algorithm?

- Gets better

How does a woman's state change over the execution of the algorithm?

- Free → engaged

What can we say about a man's partner?

- Gets worse

Algorithm Analysis

What is the running time of this algorithm?

- $O(n^2)$

What is the state complexity of this algorithm?

- $O(n^2)$

Proof of Correctness: Termination

Observation 1. Men propose to women in decreasing order of preference

Observation 2. Once a woman is matched, she never becomes unmatched; she only "trades up"

Claim. Algorithm terminates after at most n^2 iterations of while loop.

	1st	2nd	3rd	4th	5th
Victor	A	B	C	D	E
Wyatt	B	C	D	A	E
Xavier	C	D	A	B	E
Yancey	D	A	B	C	E
Zeus	A	B	C	D	E

	1st	2nd	3rd	4th	5th
Amy	W	X	Y	Z	V
Bertha	X	Y	Z	V	W
Clare	Y	Z	V	W	X
Diane	Z	V	W	X	Y
Erika	V	W	X	Y	Z

n(n-1) + 1 proposals required

Proof of Correctness: Termination

Observation 1. Men propose to women in decreasing order of preference

Observation 2. Once a woman is matched, she never becomes unmatched; she only "trades up"

Claim. Algorithm terminates after at most n2 iterations of while loop.

Pf. Each time through the while loop a man proposes to a new woman. There are only n^2 possible proposals.

- n(n-1) + 1 proposals required

Algorithm Analysis

Prove that final matching is perfect matching

If *m* is free at some point in the execution of the algorithm, then there is a woman to whom he has not yet proposed.

Claim. All men and women get matched.

Proof of Correctness: Perfection

Claim. All men and women get matched.

Pf. (by contradiction)

- Suppose that *m* is not matched upon termination of algorithm
- Then some woman, say *w*, is not matched upon termination.
- By Observation 2, *w* was never proposed to.
- But, m proposes to everyone, since he ends up unmatched (while condition)

Proof of Correctness: Stability

Claim. No unstable pairs.

S*

Amy-Yancey
Bertha-Zeus
. . .

Proof of Correctness: Stability

Claim. No unstable pairs.

S*

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. . .

Pf. (by contradiction)

- Suppose A-Z is an unstable pair: each prefers each other to partner in Gale-Shapley matching S*.
- Case 1: Z never proposed to A. ← men propose in decreasing order of preference
  - ⇒ Z prefers his GS partner to A.
  - ⇒ A-Z is stable.
- Case 2: Z proposed to A. ← women only trade up
  - ⇒ A rejected Z (right away or later)
  - ⇒ A prefers her GS partner to Z.
  - ⇒ A-Z is stable.
- In either case A-Z is stable, a contradiction. ▪