Objectives

Network flow

- Maximum flow
- Minimum cuts

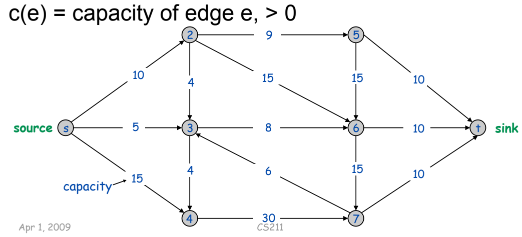
Flow Network

Abstraction for material *flowing* through the edges

G = (V, E) = directed graph, no parallel edges

Two distinguished nodes: *s* = source, *t* = sink

c(e) = capacity of edge e, > 0

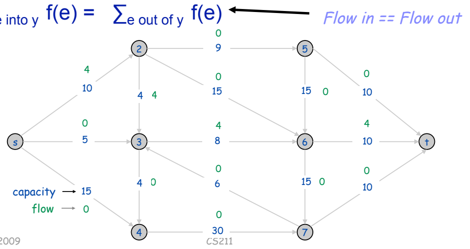


Flows

An **s-t flow** is a function that satisfies

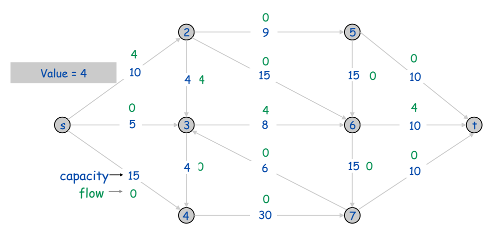
Flow can't exceed capacity

- *Capacity* condition: For each e ∈ E: $0 \le f(e) \le c(e)$
- *Conservation* condition: For each v ∈ V – {s, t}:
  $\sum_{e \text{ into } v} f(e) = \sum_{e \text{ out of } v} f(e)$ *Flow in == Flow out*



Flows

The value of a flow f is $v(f) = \sum_{e \text{ out of } s} f(e)$

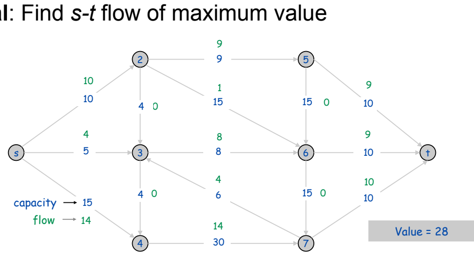


Maximum Flow Problem

Make network most efficient

- Use most of available capacity

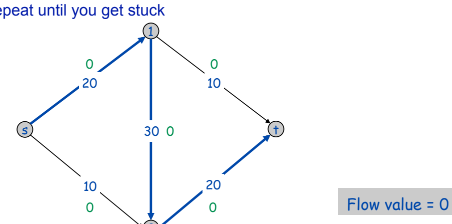
Goal: Find *s-t* flow of maximum value



Towards a Max Flow Algorithm

Greedy algorithm

- Start with f(e) = 0 for all edge e ∈ E
- Find an *s-t* path P where each edge has f(e) < c(e)
- Augment flow along path P
- Repeat until you get stuck



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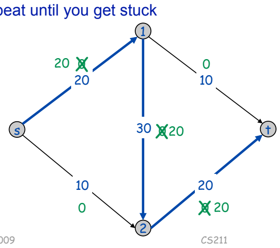

Flow value = 20

Towards a Max Flow Algorithm

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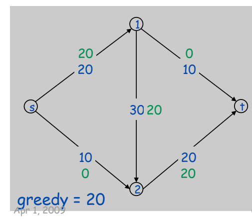
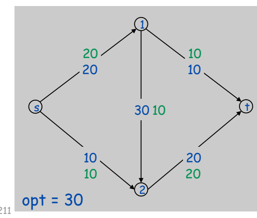
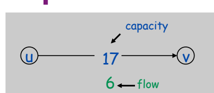
locally optimality does not ⇒ global optimality





Residual Graph

Original edge: e = (u, v) ∈ E
- Flow f(e), capacity c(e)



Residual Graph

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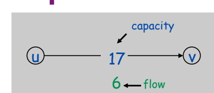
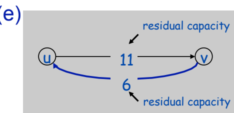


Residual edge
- e = (u, v) w/ capacity c(e) - f(e)
- e^R = (v, u) with capacity f(e)
  - To undo flow



Residual graph: $G_f = (V, E_f)$
- Residual edges with positive residual capacity
- $E_f = \underbrace{\{e : f(e) < c(e)\}}_{\text{Forward edges}} \cup \underbrace{\{e^R : f(e) > 0\}}_{\text{Backward edges}}$

Applying Residual Graph

Used to find the maximum flow
- Use similar idea to greedy algorithm

Residual path: simple *s-t* path in G_f
- Also known as *augmenting path*

Augmenting Path Algorithm

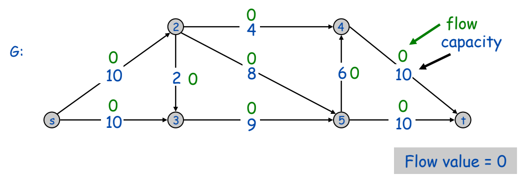
```
Ford-Fulkerson(G, s, t, c)
   foreach e ∈ E  f(e) = 0  # initially no flow
   Gf = residual graph

   while there exists augmenting path P
      f = Augment(f, c, P)     # change the flow
      update Gf             # build a new residual graph

   return f
```

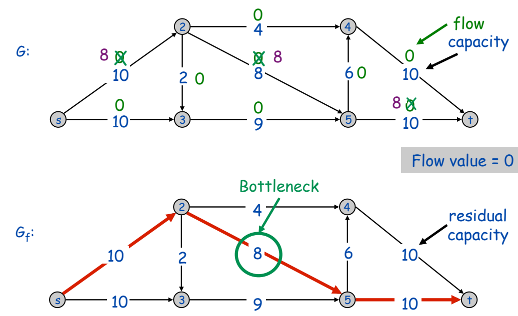
```
Augment(f, c, P)
   b = bottleneck(P) # edge on P with least capacity
   foreach e ∈ P
      if (e ∈ E) f(e) = f(e) + b  # forward edge, ↑ flow
      else       f(eR) = f(e) - b  # forward edge, ↓ flow
   return f
```

Ford-Fulkerson Algorithm

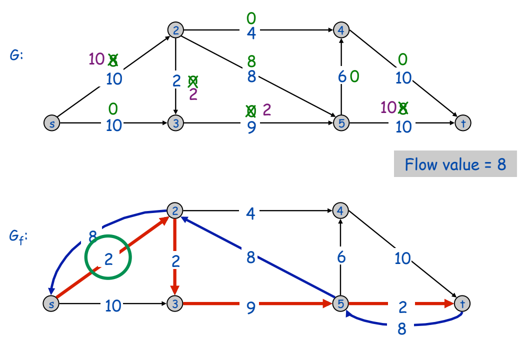


What does the residual graph look like?

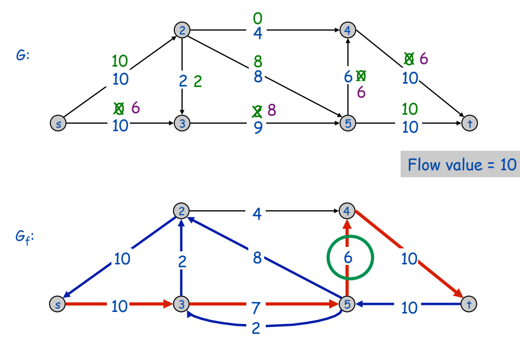
Ford-Fulkerson Algorithm



Ford-Fulkerson Algorithm



Ford-Fulkerson Algorithm



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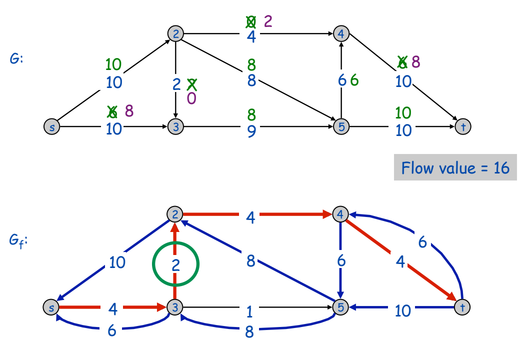

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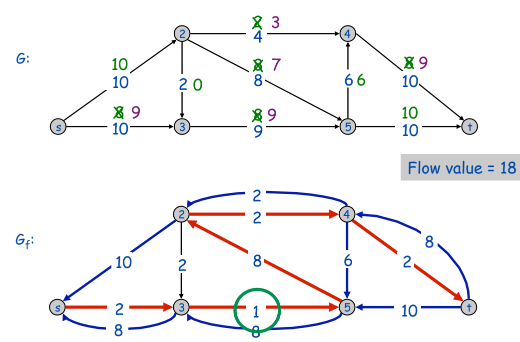

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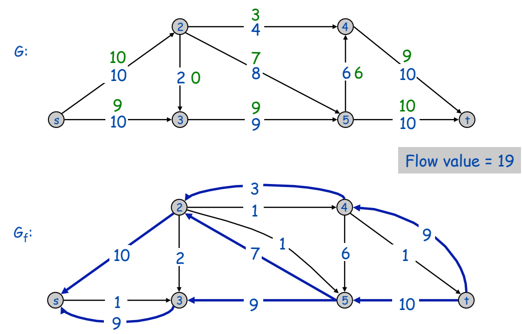
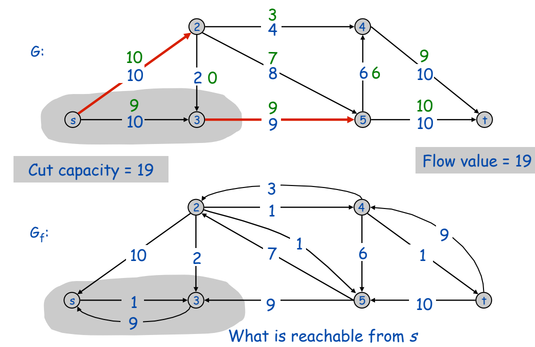


Ford-Fulkerson Algorithm



Analyzing Algorithm

Why does this algorithm work?

What is happening at each iteration?

Analyzing Augmenting Path Algorithm

```
Ford-Fulkerson(G, s, t, c)
   foreach e ∈ E  f(e) = 0  # initially no flow
   Gf = residual graph

   while there exists augmenting path P
      f = Augment(f, c, P)      # change the flow
      update Gf               # build a new residual graph

   return f
```

```
Augment(f, c, P)
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      else       f(eR) = f(e) - b  # forward edge, ↓ flow
   return f
```


Analyzing Augmenting Path Algorithm

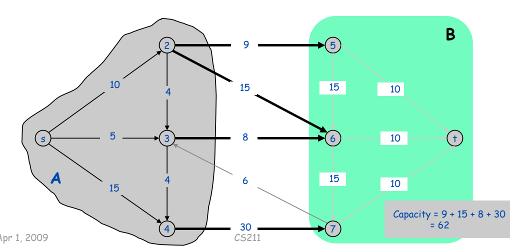
```
     Ford-Fulkerson(G, s, t, c)
O(m)    foreach e ∈ E  f(e) = 0  # initially no flow
O(m)    Gf = residual graph
Find path: O(m);  Iterations: O(C) iterations, where C = max flow
        while there exists augmenting path P
O(m)       f = Augment(f, c, P)      # change the flow
O(m)       update Gf               # build a new residual graph

        return f           Total: O(Cm)
```

```
     Augment(f, c, P)
O(n)  b = bottleneck(P) # edge on P with least capacity
O(n)  foreach e ∈ P
O(1)     if (e ∈ E) f(e) = f(e) + b  # forward edge, ↑ flow
O(1)     else       f(eR) = f(e) - b  # forward edge, ↓ flow
      return f
                   Total: O(n) → O(m), since n ≤ 2m
```


Cuts

An **s-t cut** is a partition (A, B) of V with s ∈ A and t ∈ B

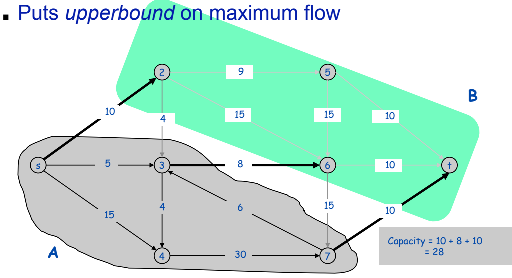
The **capacity** of a cut (A, B) is $cap(A, B) = \sum_{e \text{ out of } A} c(e)$



Minimum Cut Problem

Find an s-t cut of *minimum* capacity

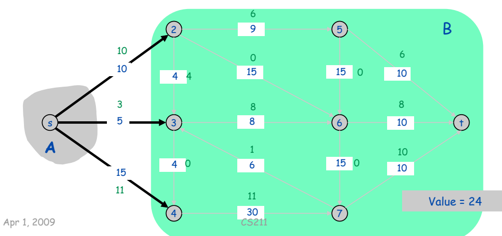
- Puts *upperbound* on maximum flow



Flow Value Lemma

Let f be any flow, and let (A, B) be any s-t cut. Then, the ***net flow*** sent across the cut is equal to the amount leaving s.

$$\sum_{e \text{ out of } A} f(e) - \sum_{e \text{ in to } A} f(e) = v(f)$$



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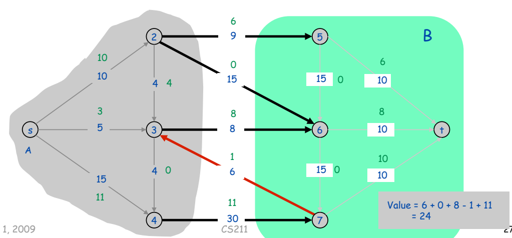

Flow Value Lemma

Let f be any flow, and let (A, B) be any s-t cut.

Then $\sum_{e \text{ out of } A} f(e) - \sum_{e \text{ in to } A} f(e) = v(f).$

Pf. By definition $v(f) = \sum_{e \text{ out of } s} f(e)$

by flow conservation, all terms except v = s are 0 $= \sum_{v \in A} \left(\sum_{e \text{ out of } v} f(e) - \sum_{e \text{ in to } v} f(e) \right)$

Possibilities for edge e:
- Both ends in A (0)
- Points out from A (+)
- Points in to A (-)

$= \sum_{e \text{ out of } A} f(e) - \sum_{e \text{ in to } A} f(e).$