

Objectives

Reviewing Presentations
Stable Matching Problem

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Road to a Dissertation and Beyond...

Course work beyond master's degree
Research proposal

- ~2 papers accepted or ready for submission

Dissertation

- Culmination of 4 papers
- Tell a story

After the dissertation

- A few more publications directly from the dissertation
- Expand

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ALGORITHMS

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Matching Residents to Hospitals

Goal: Given a set of preferences among hospitals and medical school students, design a **self-reinforcing** admissions process.

Unstable pair: applicant x and hospital y are unstable if:

- x prefers y to its assigned hospital
- y prefers x to one of its admitted students

Stable assignment: Assignment with no unstable pairs

- Natural and desirable condition
- Individual self-interest will prevent any applicant/hospital deal from being made

What details make this problem tricky?

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Stable Matching Problem

Simplified version of resident-matching problem

Goal: Given n men and n women, find a "suitable" matching

- Participants rate members of opposite sex
- Each man lists women in order of preference from best to worst
- Each woman lists men in order of preference from best to worst

| | | | | | | | |
|---------------------------------|-----------------|-----------------|-----------------|-----------------------------------|------------------|-----------------|-----------------|
| | favorite ↓ | | | | least favorite ↓ | | |
| | 1 st | 2 nd | 3 rd | | 1 st | 2 nd | 3 rd |
| Xavier | Amy | Bertha | Clare | Amy | Yancey | Xavier | Zeus |
| Yancey | Bertha | Amy | Clare | Bertha | Xavier | Yancey | Zeus |
| Zeus | Amy | Bertha | Clare | Clare | Xavier | Yancey | Zeus |
| <i>Men's Preference Profile</i> | | | | <i>Women's Preference Profile</i> | | | |

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Stable Matching Problem

Perfect matching: everyone is matched monogamously

- Each man is paired with exactly one woman
- Each woman is paired with exactly one man

Stability: no incentive for some pair of participants to undermine assignment by joint action

- In matching M , an *unmatched* pair m - w is unstable if man m and woman w prefer each other to current partners
- Unstable pair m - w could each improve by eloping

Stable matching: perfect matching with no unstable pairs

Stable matching problem. Given the preference lists of n men and n women, find a stable matching if one exists.

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Any Questions?

What are you wondering about this problem at this point?

- Is it possible to match everyone?
- Can we be fair in the matching? (preferences)
- Will the matching always be the same?

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Propose-And-Reject Algorithm

[Gale-Shapley 1962]

Intuitive method that guarantees to find a stable matching

```

Initialize each person to be free.
while (some man is free and hasn't proposed to every woman) {
  Choose such a man m
  w = 1st woman on m's list to whom m has not yet proposed
  if (w is free)
    assign m and w to be engaged
  else if (w prefers m to her fiancé m')
    assign m and w to be engaged, and m' to be free
  else
    w rejects m
}
    
```

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Observations about the Algorithm

What can we say about any woman's partner over the execution of the algorithm?

- Gets better

How does a woman's state change over the execution of the algorithm?

- Free → engaged

What can we say about a man's partner?

- Gets worse

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Algorithm Analysis

What is the running time of this algorithm?

- $O(n^2)$

What is the state complexity of this algorithm?

- $O(n^2)$

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Proof of Correctness: Termination

Observation 1. Men propose to women in decreasing order of preference

Observation 2. Once a woman is matched, she never becomes unmatched; she only "trades up"

Claim. Algorithm terminates after at most n^2 iterations of while loop.

| | 1 st | 2 nd | 3 rd | 4 th | 5 th | | 1 st | 2 nd | 3 rd | 4 th | 5 th |
|--------|-----------------|-----------------|-----------------|-----------------|-----------------|--------|-----------------|-----------------|-----------------|-----------------|-----------------|
| Victor | A | B | C | D | E | Amy | W | X | Y | Z | V |
| Wyatt | B | C | D | A | E | Bartha | X | Y | Z | V | W |
| Xavier | C | D | A | B | E | Clare | Y | Z | V | W | X |
| Yancy | D | A | B | C | E | Diane | Z | V | W | X | Y |
| Zoe | A | B | C | D | E | Enka | V | W | X | Y | Z |

$n(n-1) + 1$ proposals required

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Proof of Correctness: Termination

Observation 1. Men propose to women in decreasing order of preference

Observation 2. Once a woman is matched, she never becomes unmatched; she only "trades up"

Claim. Algorithm terminates after at most n^2 iterations of while loop.

Pf. Each time through the while loop a man proposes to a new woman. There are only n^2 possible proposals.

- $n(n-1) + 1$ proposals required

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Algorithm Analysis

Prove that final matching is perfect matching

If m is free at some point in the execution of the algorithm, then there is a woman to whom he has not yet proposed.

Claim. All men and women get matched.

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Proof of Correctness: Perfection

Claim. All men and women get matched.

Pf. (by contradiction)

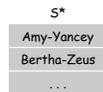
- Suppose that m is not matched upon termination of algorithm
- Then some woman, say w , is not matched upon termination.
- By Observation 2, w was never proposed to.
- But, m proposes to everyone, since he ends up unmatched (while condition)

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Proof of Correctness: Stability

Claim. No unstable pairs.



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Proof of Correctness: Stability

Claim. No unstable pairs.



Pf. (by contradiction)

- Suppose A-Z is an unstable pair: each prefers each other to partner in Gale-Shapley matching S^* .
- Case 1: Z never proposed to A. ← men propose in decreasing order of preference
 ⇒ Z prefers his GS partner to A.
 ⇒ A-Z is stable.
- Case 2: Z proposed to A. ← women only trade up
 ⇒ A rejected Z (right away or later)
 ⇒ A prefers her GS partner to Z.
 ⇒ A-Z is stable.
- In either case A-Z is stable, a contradiction. ▪

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