Objectives

- Greedy Algorithms
  - Interval partitioning
  - Minimizing Lateness
- Greedy stays ahead
- Exchange argument

Review: Greedy Algorithm Template

- Consider jobs (or whatever) in some order
  - Decision: What order is best?
- Take each job provided it's compatible with the ones already taken

Greedy Algorithms

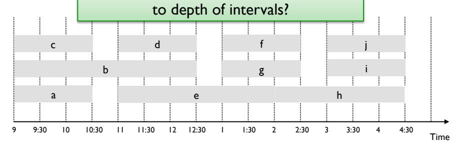
- At each step, take as much as you can get
  - Feasible – satisfy problem's constraints
  - Locally optimal – best local choice among available feasible choices
  - Irrevocable – after decided, no going back

Interval Partitioning: Lower Bound on Optimal Solution

- Def. The depth of a set of open intervals is the maximum number that contain any given time.
- Key observation. # of classrooms needed ≥ depth.
- Ex: Depth of schedule below = 3 ⇒ schedule below is optimal.

a, b, c all contain 9:30

Does there always exist a schedule equal to depth of intervals?



Interval Partitioning: Greedy Algorithm

- Consider lectures in increasing order of start time: assign lecture to any compatible classroom

```
Sort intervals by starting time so that s1 ≤ s2 ≤ ... ≤ sn
d = 0   <-- number of allocated classrooms
for j = 1 to n
   if (lecture j is compatible with some classroom k)
      schedule lecture j in classroom k
   else
      allocate a new classroom d + 1
      schedule lecture j in classroom d + 1
      d = d + 1
```

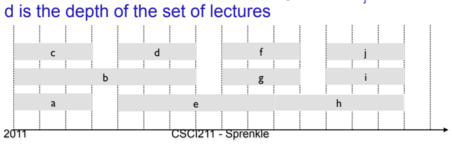
- Implementation: O(n log n)
  - For each classroom k, maintain the finish time of the last job added.
  - Keep the classrooms in a priority queue by last job finish time.

Interval Partitioning: Greedy Analysis

- Observation. Greedy algorithm never schedules two incompatible lectures in the same classroom
- Theorem. Greedy algorithm is optimal
- Pf Intuition
  - When do we add more classrooms?
  - When would we add the d+1 classroom?

Interval Partitioning: Greedy Analysis

- Observation. Greedy algorithm never schedules two incompatible lectures in the same classroom
- Theorem. Greedy algorithm is optimal
- Pf.
  - Let d = number of classrooms that greedy algorithm allocates
  - Classroom d is opened because we needed to schedule a job, say j, that is incompatible with all d-1 other classrooms
  - Since we sorted by start time, all these incompatibilities are caused by lectures that start no later than s_j
  - Thus, we have d lectures overlapping at time $s_j + \varepsilon$
  - d is the depth of the set of lectures



Proving Greedy Algorithms Work

- Specifically, produce an **optimal** solution
- Approaches:
  - Greedy algorithm stays ahead
    - Does better than any other algorithm at each step
  - Exchange argument
    - Transform any solution into a greedy solution
  - Structural Argument
    - Figure out some structural bound that all solutions must meet

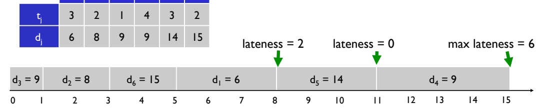
Exchange argument

SCHEDULING TO MINIMIZE LATENESS

Scheduling to Minimizing Lateness

- Single resource processes one job at a time
- Job j requires t_j units of processing time and is due at time d_j (its deadline)
- If j starts at time s_j, it finishes at time $f_j = s_j + t_j$
- Lateness: $\ell_j = \max \{ 0, f_j - d_j \}$
- **Goal**: schedule all jobs to *minimize* **maximum lateness** $L = \max \ell_j$

	1	2	3	4	5	6
t_j	3	2	1	4	3	2
d_j	6	8	9	9	14	15



Note: **not** a sum total

Developing Greedy Algorithms

- What do we want to optimize?
- What order?
  - Intuition of order?
  - Counter examples for order being optimal?

Minimizing Lateness: Possible Orderings

- Shortest processing time first. Consider jobs in ascending order of processing time t_j.

Counter example

	1	2
t_j	1	10
d_j	100	10

- Smallest slack. Consider jobs in ascending order of slack $d_j - t_j$.

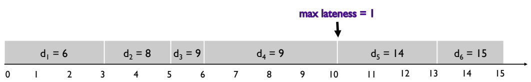
Counter example

	1	2
t_j	1	10
d_j	2	10

Minimizing Lateness: Greedy Algorithm

- Earliest deadline first.

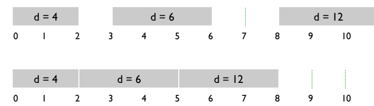
```
Sort n jobs by deadline so that d1 ≤ d2 ≤ … ≤ dn
t = 0
for j = 1 to n
   Assign job j to interval [t, t + tj]
   sj = t
   fj = t + tj
   t = t + tj
output intervals [sj, fj]
```



What can we say about this algorithm/its results?

Minimizing Lateness: No Idle Time

- Observation. There exists an optimal schedule with no idle time



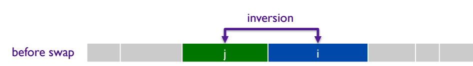
- Observation. The greedy schedule has no idle time

Proving Optimality

- Goal: Prove greedy algorithm produces optimal solution
- Approach: **Exchange argument**
  - Start with an optimal schedule Opt
  - Gradually modify Opt
    - Preserving its optimality
  - Transform into a schedule identical to greedy's schedule

Minimizing Lateness: Inversions

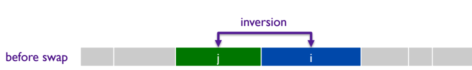
- Def. An ***inversion*** in schedule S is a pair of jobs i and j such that: $d_i < d_j$ but j scheduled before i



Can Greedy's solution have any inversions?

Minimizing Lateness: Inversions

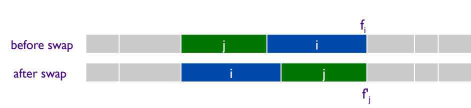
- Def. An ***inversion*** in schedule S is a pair of jobs i and j such that: $d_i < d_j$ but j scheduled before i



Greedy's schedule has no inversions!

Minimizing Lateness: Inversions

- Claim. Swapping two adjacent jobs with the same deadline does not increase the max lateness
- Pf Sketch. Let ℓ be the lateness before the swap, and let ℓ' be it afterwards
  - Lateness of other jobs?
  - Lateness of i? j?

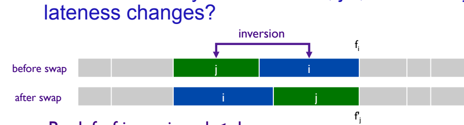


Minimizing Lateness: Inversions

- Claim. Swapping two adjacent jobs with the same deadline does not increase the max lateness
- Pf. Let ℓ be the lateness before the swap, and let ℓ' be it afterwards
  - Lateness remains the same for all other jobs:
    - $\ell'_k = \ell_k$ for all $k \neq i, j$
  - Lateness of i before is $f_i - d_i = t_i + t_j - d_i$
  - Lateness of j after is $f_j' - d_j = t_i + t_j - d_j$
    - But $d_i = d_j$

Minimizing Lateness: Inversions

- Claim. Swapping two adjacent, inverted jobs reduces the number of inversions by one and does *not increase the max lateness*
  - How do we know inversions are adjacent?
- Pf Setup. Let ℓ be the lateness before the swap, and let ℓ' be it afterwards
  - What can we say about how i's, j's, and other jobs' lateness changes?



By def of inversion, $d_i < d_j$

Minimizing Lateness: Inversions

- Claim. Swapping two adjacent, inverted jobs reduces the number of inversions by one and does *not increase the max lateness*.
- Pf. Let ℓ be the lateness before the swap, and let ℓ' be it afterwards
  - $\ell'_k = \ell_k$ for all $k \neq i, j$
  - $\ell'_i \leq \ell_i$
  - If job j is late:

$$\begin{aligned} \ell'_j &= f'_j - d_j && \text{(definition)} \\ &= f_i - d_j && (j \text{ finishes at time } f_i) \\ &\leq f_i - d_i && (i < j) \\ &\leq \ell_i && \text{(definition)} \end{aligned}$$

Greedy Analysis Strategies

- Greedy algorithm stays ahead. Show that after each step of the greedy algorithm, its solution is at least as good as any other algorithm's.
- Exchange argument. Gradually transform any solution to the one found by the greedy algorithm without hurting its quality.
- Structural. Discover a simple "structural" bound asserting that every possible solution must have a certain value. Then show that your algorithm always achieves this bound.

PS2

- Make clear the input to an algorithm
  - Don't want me guessing as to what you're doing because I might be wrong
- Always analyze the running time of your algorithms
  - Whether stated in problem or not
- Comparison of runtimes

Assignments

- Exam 1
  - Open book, open notes, open lecture notes
  - **NO OTHER RESOURCES**
  - I mention explicitly to analyze your algorithms' running times. I will not do that in the future.

Minimizing Lateness: Analysis of Greedy Algorithm

- Theorem. Greedy schedule S is optimal
- Pf idea. Convert Opt to Greedy
  - Does opt schedule have idle time?
  - What if opt schedule has no inversions?
  - What if opt schedule has inversions?

Minimizing Lateness: Analysis of Greedy Algorithm

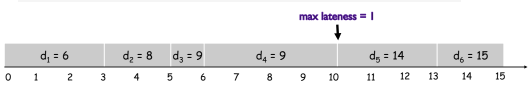
- Theorem. Greedy schedule S is optimal
- Pf. Define S* to be an optimal schedule that has the fewest number of inversions, and let's see what happens
  - Can assume S* has no idle time
  - If S* has no inversions, then S = S*
  - If S* has an inversion, let i-j be an adjacent inversion
    - Swapping i and j does not increase the maximum lateness and strictly decreases the number of inversions
    - This contradicts definition of S* ▪

Analyzing Running Time

- Earliest deadline first.

```
Sort n jobs by deadline so that d1 ≤ d2 ≤ … ≤ dn
t = 0
for j = 1 to n
   Assign job j to interval [t, t + tj]
   sj = t
   fj = t + tj
   t = t + tj
output intervals [sj, fj]
```

O(n logn)



What is the runtime of this algorithm?

Greedy Exchange Proofs

1. Label your algorithm's solution and a general solution.
   - Example: let A = $\{a_1, a_2, ..., a_k\}$ be the solution generated by your algorithm, and let O = $\{o_1, o_2, ..., o_m\}$ be an arbitrary (or optimal) feasible solution.
2. Compare greedy with other solution.
   - Assume that your arbitrary/optimal solution is not the same as your greedy solution (since otherwise, you are done).
   - Typically, can isolate a simple example of this difference, such as:
     1. There is an element e ∈ O that ∉ A and an element f ∈ A that ∉ O
     2. 2 consecutive elements in O are in a different order than in A (i.e., there is an *inversion*).
3. Exchange.
   - Swap the elements in question in O (either ① swap one element out and another in or ② swap the order of the elements) and argue that solution is no worse than before.
   - Argue that if you continue swapping, you eliminate all differences between O and A in a *finite* # of steps without worsening the solution's quality.
   - Thus, the greedy solution produced is just as good as any optimal solution, and hence is optimal itself.