Objectives

- Data structure: Heaps
- Implementing a Priority Queue

Review: Priority Queues for Sorting

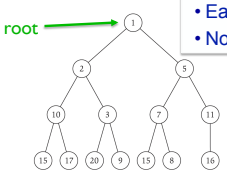
1. Add elements into PQ with the number's value as its priority
2. Then extract the smallest number until done
   - Come out in sorted order

Sorting *n* numbers takes O(n logn) time, which is our goal running time. However, "known" data structures won't give us that running time.

Already know our "loops" will be O(n)

Heap Defined

- Combines benefits of sorted array and list
- Balanced binary tree

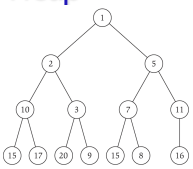


- Each node has *at most* 2 children
- Node value is its *key*

Heap order: each node's key is at least as large as its parent's

Note: **not** a binary search tree

Review: Implementing a Heap



- Option 1: Use pointers
  - Each node keeps
    - Element it stores, key
    - 3 pointers: 2 children, parent
- Option 2: No pointers
  - Requires knowing upper bound on *n*
  - For node at position *i*
    - left child is at *2i*
    - right child is at *2i+1*

Implementing a Heap: Operations

- Finding the minimal element?

Implementing a Heap: Operations

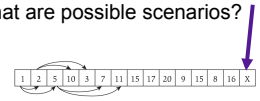
- Finding the minimal element
  - First element
  - O(1)

Implementing a Heap: Operations

- Adding an element?
  - Assume heap has less than N elements

Implementing a Heap: Operations

- Adding an element?
  - Could add element to last position
    - What are possible scenarios?



Implementing a Heap: Operations

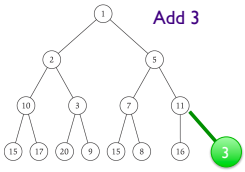
- Adding an element?
  - Could add element to last position
    - What are possible scenarios?
      - Heap is no longer balanced
      - Something that is almost a heap but a little off
      - Need `Heapify-up` procedure to fix our heap

Heapify-Up

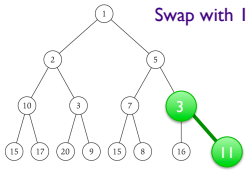
Heap — Position where node added

```
Heapify-up(H, i):
  if i > 1 then
      j=parent(i)=floor(i/2)
      if key[H[i]] < key[H[j]] then
        swap array entries H[i] and H[j]
        Heapify-up(H, j)
```


Practice: Heapify-Up



Practice: Heapify-Up



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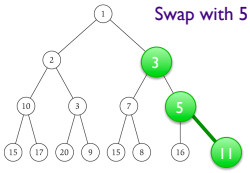

Heapify-Up

- Claim. Assuming array H is almost a heap with key of H[i] too small, Heapify-Up fixes the heap property in O(log i) time
  - Can insert a new element in a heap of *n* elements in O(log n) time

Heapify-Up

- Claim. Assuming array H is almost a heap with key of H[i] too small, Heapify-Up fixes the heap property in O(log i) time
  - Can insert a new element in a heap of *n* elements in O(log n) time
- Proof. By induction
  - If i=1 …

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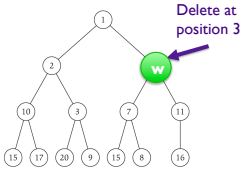
  - Can insert a new element in a heap of *n* elements in O(log n) time
- Proof. By induction
  - If i=1, is already a heap → O(1)
  - If i>1, …

Heapify-Up

- Claim. Assuming array H is almost a heap with key of H[i] too small, Heapify-Up fixes the heap property in O(log i) time
  - Can insert a new element in a heap of *n* elements in O(log n) time
- Proof. By induction
  - If i=1, is already a heap → O(1)
  - If i>1,
    - Swaps are O(1)
    - Swaps continue up to root (max) → log i

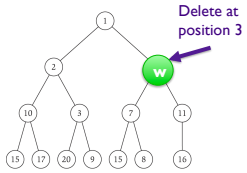
Deleting an Element



Deleting an Element

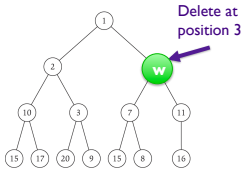
- Delete at position *i*
- Removing an element:
  - Messes up heap order
  - Leaves a "hole" in the heap
- Not as straightforward as `Heapify-Up`
- Algorithm
  1. Fill in element where hole was
     - Patch hole: move n^{th} element into i^{th} spot
  2. Adjust heap to be in order
     - At position *i* because moved n^{th} item up to *i*

Deleting an Element



- What are the possibilities when we move n^{th} element (*w*) into spot where element was removed?

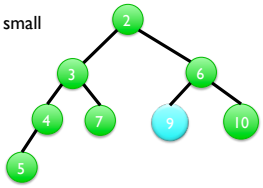
Deleting an Element



- Two possibilities: element *w* is
  - Too small: violation is between it and parent → `Heapify-Up`
  - Too big: with one or both children → `Heapify-Down` (example: w = 12)

Deleting an Element

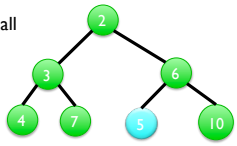
Example where new key is too small



- Delete 9
- Replace with 5

Deleting an Element

Example where new key is too small



- Delete 9
- Replace with 5
- But 5 < 6, so need to `Heapify-Up`

Heapify-Down

```
Heapify-down(H, i):
   n = length(H)
   if 2i > n then          Why can we stop?
      Terminate with H unchanged
   else if 2i < n then
      left=2i and right=2i+1
      j be index that minimizes
            key[H[left]] and key[[H[right]]
   else if 2i = n then
      j=2i

   if key[H[j]] < key[H[i]] then
      swap array entries H[i] and H[j]
      Heapify-down(H, j)
```

Heapify-Down

```
Heapify-down(H, i):
   n = length(H)
   if 2i > n then          i is a leaf – nowhere to go
      Terminate with H unchanged
   else if 2i < n then
      left=2i and right=2i+1
      j be index that minimizes
            key[H[left]] and key[[H[right]]
   else if 2i = n then
      j=2i

   if key[H[j]] < key[H[i]] then
      swap array entries H[i] and H[j]
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```
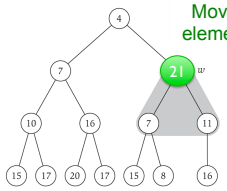
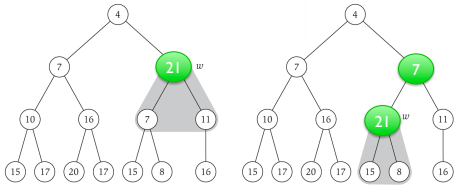


Practice: Heapify-Down

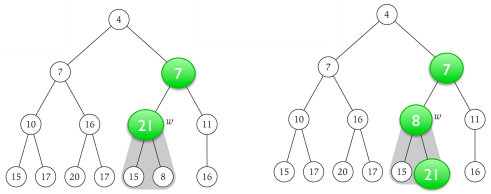
Moved 21 to where element was removed



Practice: Heapify-Down



Practice: Heapify-Down



Runtime of Heapify-Down?

```
Heapify-down(H, i):
   n = length(H)
   if 2i > n then
      Terminate with H unchanged
   else if 2i < n then
      left=2i and right=2i+1
      j be index that minimizes    O(1)
            key[H[left]] and key[[H[right]]
   else if 2i = n then
      j=2i

   if key[H[j]] < key[H[i]] then
      swap array entries H[i] and H[j] O(1)
      Heapify-down(H, j)
```

Num swaps: O(log n)

Implementing Priority Queues with Heaps

Operation	Description	Run Time
StartHeap(N)	Creates an empty heap that can hold N elements	
Insert(v)	Inserts item v into heap	
FindMin()	Identifies minimum element in heap but does not remove it	
Delete(i)	Deletes element in heap at position i	
ExtractMin()	Identifies and deletes an element with minimum key from heap	

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Comparing Data Structures

Operation	Heap	Unsorted List	Sorted List
StartHeap(N)			
Insert(v)			
FindMin()			
Delete(i)			
ExtractMin()			

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Comparing Data Structures

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Additional Heap Operations

- Access elements in PQ by name
  - Maintain additional array `Position` that stores current position of each element in heap

- Operations:
  - Delete(Position[v])
    - Does not increase overall running time
  - ChangeKey(v, α)
    - Changes key of element v to key(v) = α
    - Identify position of element v in array (`Position` array)
    - Change key, heapify

Assignments

- Journals: Finish Chapter 2 for Wednesday
- Problem Set 2 due Friday