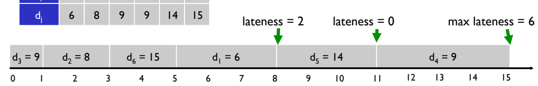
Objectives

- Wrap Up: Minimizing Lateness
  - Greedy exchange
- Problem: Shortest Path

Review: Scheduling to Minimizing Lateness

- Single resource processes one job at a time
- Job j requires t_j units of processing time and is due at time d_j (its deadline)
- If j starts at time s_j, it finishes at time $f_j = s_j + t_j$
- Lateness: $\ell_j = \max \{ 0, f_j - d_j \}$
- **Goal**: schedule all jobs to *minimize* **maximum lateness** $L = \max \ell_j$

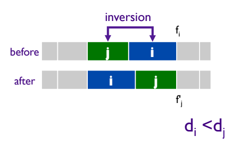
	1	2	3	4	5	6
t_j	3	2	1	4	3	2
d_j	6	8	9	9	14	15



Note: **not** a sum total

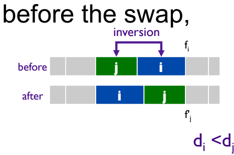
Minimizing Lateness: Inversions

- Claim. Swapping two adjacent, inverted jobs reduces the number of inversions by one and does *not increase the max lateness*.
- How to prove?



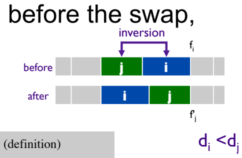
Minimizing Lateness: Inversions

- Claim. Swapping two adjacent, inverted jobs reduces the number of inversions by one and does *not increase the max lateness*.
- Pf. Let ℓ be the lateness before the swap, and let ℓ' be it afterwards



Minimizing Lateness: Inversions

- Claim. Swapping two adjacent, inverted jobs reduces the number of inversions by one and does *not increase the max lateness*.
- Pf. Let ℓ be the lateness before the swap, and let ℓ' be it afterwards
  - $\ell'_k = \ell_k$ for all $k \neq i, j$
  - $\ell_j \leq \ell_i$, $\ell'_i \leq \ell_i$
  - If job j is late:



$$\begin{aligned} \ell'_j &= f'_j - d_j && \text{(definition)} \\ &= f_i - d_j && (j \text{ finishes at time } f_i) \\ &\leq f_i - d_i && (i < j) \\ &\leq \ell_i && \text{(definition)} \end{aligned}$$

Shows that the lateness of jobs i and j do not increase from the original order

Minimizing Lateness: Analysis of Greedy Algorithm

- Theorem. Greedy schedule S is optimal
- Pf idea. Convert Opt to Greedy
  - Does opt schedule have idle time?
  - What if opt schedule has no inversions?
  - What if opt schedule has inversions?

Minimizing Lateness: Analysis of Greedy Algorithm

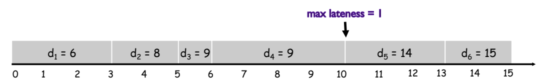
- Theorem. Greedy schedule S is optimal
- Pf. Define S* to be an optimal schedule that has the fewest number of inversions, and let's see what happens
  - Can assume S* has no idle time
  - If S* has no inversions (and no idle time), then S = S*
  - If S* has an inversion, let i-j be an adjacent inversion
    - Swapping i and j does not increase the maximum lateness and strictly decreases the number of inversions
    - This contradicts definition of S* ▪

Analyzing Running Time

- Earliest deadline first.

```
Sort n jobs by deadline so that d1 ≤ d2 ≤ … ≤ dn
t = 0
for j = 1 to n
   Assign job j to interval [t, t + tj]
   sj = t
   fj = t + tj
   t = t + tj
output intervals [sj, fj]
```

O(n logn)



What is the runtime of this algorithm?

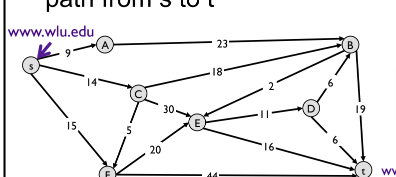
Greedy Exchange Proofs

1. Label your algorithm's solution and a general solution.
   - Example: let A = $\{a_1, a_2, ..., a_k\}$ be the solution generated by your algorithm, and let O = $\{o_1, o_2, ..., o_m\}$ be an arbitrary (or optimal) feasible solution.
2. Compare greedy with other solution.
   - Assume that your arbitrary/optimal solution is not the same as your greedy solution (since otherwise, you are done).
   - Typically, can isolate a simple example of this difference, such as:
     1. There is an element $e \in O$ that $\notin A$ and an element $f \in A$ that $\notin O$
     2. 2 consecutive elements in O are in a different order than in A (i.e., there is an *inversion*).
3. Exchange.
   - Swap the elements in question in O (either ① swap one element out and another in or ② swap the order of the elements) and argue that solution is no worse than before.
   - Argue that if you continue swapping, you eliminate all differences between O and A in a *finite* # of steps without worsening the solution's quality.
   - Thus, the greedy solution produced is just as good as any optimal solution, and hence is optimal itself.

SHORTEST PATH

Shortest Path Problem

- Given
  - Directed graph G = (V, E)
  - Source s, destination t
  - Length ℓ_e = length of edge e (non-negative)
- Shortest path problem: find shortest directed path from s to t

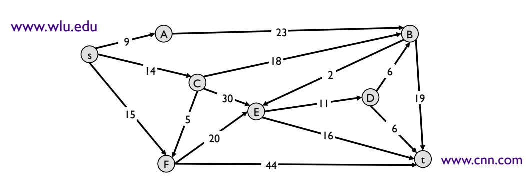


cost of path = sum of edge costs in path

Cost of path s-A-B-E-t
= 9 + 23 + 2 + 16
= 48

Shortest Path Problem

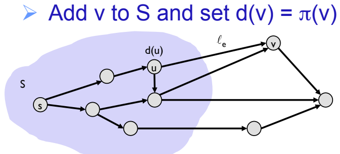
- Shortest path problem: find shortest directed path from s to t
- Towards algorithm ideas:
  - What is shortest path from s→A? s→C?
  - What is the shortest path from s→B? E? D?



Dijkstra's Algorithm

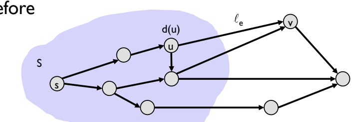
1. Maintain a set of **explored nodes** S
   - Keep the shortest path distance d(u) from *s* to *u*
2. Initialize S={s}, d(s)=0, ∀u≠s, d(u)=∞
3. Repeatedly choose unexplored node *v* which minimizes $\pi(v) = \min_{e=(u,v):u \in S} d(u) + \ell_e,$
   - Add v to S and set d(v) = π(v)

shortest path to some u in explored part, followed by a single edge (u, v)

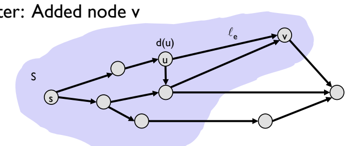


Dijkstra's Algorithm

Before



After: Added node v



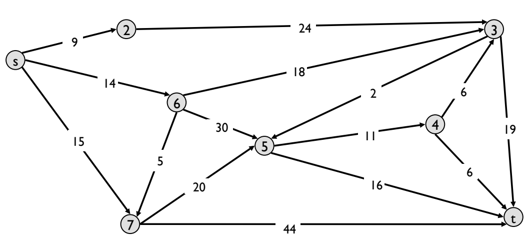
How Greedy?

How Greedy?

- We always form **shortest new *s-v* path** from a path in S followed by a *single* edge
- Proof of optimality: *Stays ahead* of all other solutions
  - Each time selects a path to a node *v*, that path is shorter than every other possible path to *v*

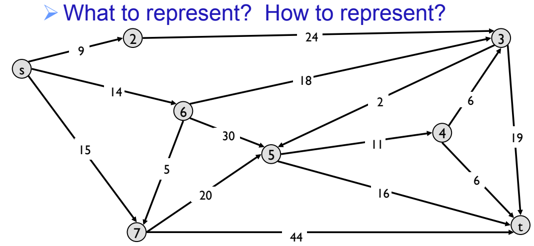
Dijkstra's Shortest Path Algorithm

- Find shortest path from s to t



Dijkstra's Shortest Path Algorithm

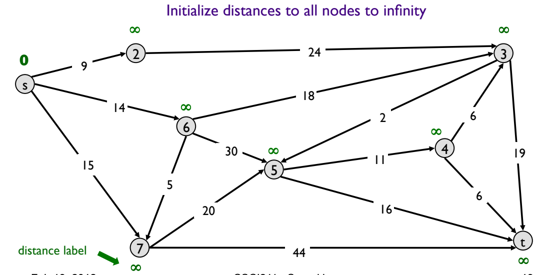
- Find shortest path from s to t
- Implementation ideas?
  - What to represent? How to represent?



Dijkstra's Shortest Path Algorithm

S = { }
PQ = { s, 2, 3, 4, 5, 6, 7, t }

Initialize distances to all nodes to infinity



Dijkstra's Shortest Path Algorithm

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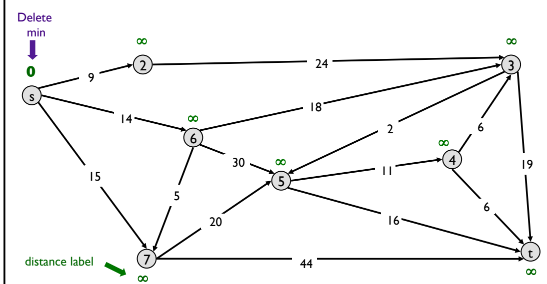
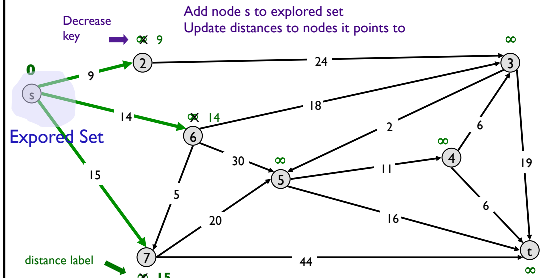


Dijkstra's Shortest Path Algorithm

S = { s }
PQ = { 2, 3, 4, 5, 6, 7, t }

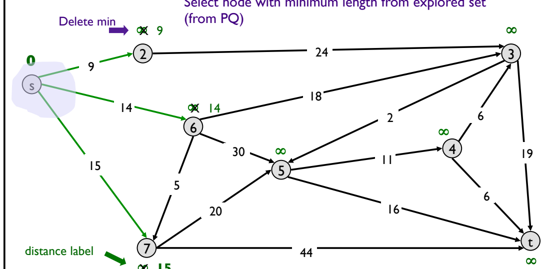
Add node s to explored set
Update distances to nodes it points to



Dijkstra's Shortest Path Algorithm

S = { s }
PQ = { 2, 6, 7, 3, 4, 5, t }

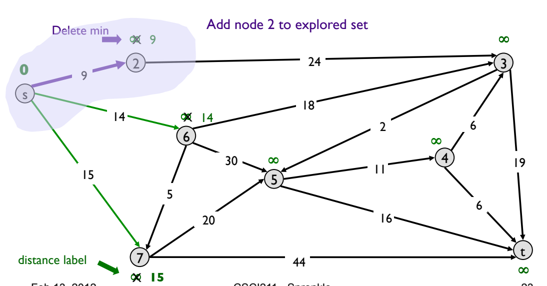
Select node with minimum length from explored set
(from PQ)



Dijkstra's Shortest Path Algorithm

S = { s, 2 }
PQ = { 6, 7, 3, 4, 5, t }

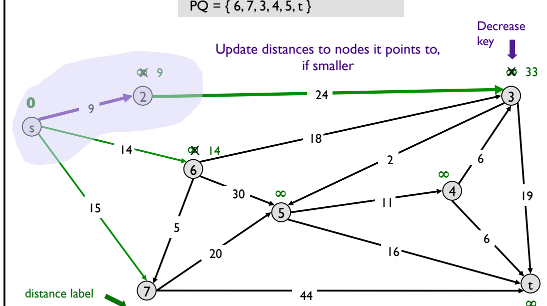
Add node 2 to explored set



Dijkstra's Shortest Path Algorithm

S = { s, 2 }
PQ = { 6, 7, 3, 4, 5, t }

Update distances to nodes it points to,
if smaller



Looking Ahead

- Exam due today at 4:30 p.m.
- Wiki due Wednesday for sections 3.5-3.6
  - Directed graphs, topological order
- PS4 due Friday