

Objectives

- Network Flow Applications
 - Bipartite matching
 - Circulation
 - Survey design
 - Airline scheduling

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Review

- What is a flow network?
- What is our usual goal given a flow network?
 - How do we reach that goal?
- What is the Ford-Fulkerson algorithm?
- What is the min-cut?
 - How does it relate to the max flow?

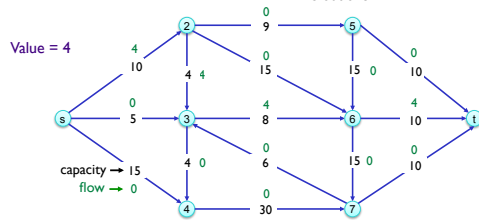
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Review: Network Flows

- An **s-t flow** is a function that satisfies
 - **Capacity condition:** For each $e \in E$: $0 \leq f(e) \leq c(e)$
 - **Conservation condition:** For each $v \in V - \{s, t\}$: $\sum_{e \text{ into } v} f(e) = \sum_{e \text{ out of } v} f(e)$
- The **value** of a flow f is $v(f) = \sum_{e \text{ out of } s} f(e)$



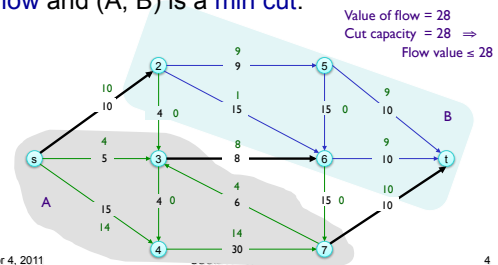
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Review: Certificate of Optimality

- **Corollary.** Let f be any flow, and let (A, B) be any cut. If $v(f) = \text{cap}(A, B)$, then f is a **max flow** and (A, B) is a **min cut**.



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Power of Max Flow Problem

Some problems with non-trivial combinatorial searches can be formulated as **max flow** or **min cut** in a directed graph

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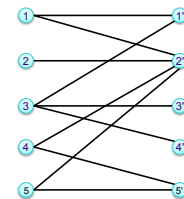
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Review: Bipartite Graph: Max Flow Formulation

Problem: find matching of largest possible size

How did we turn this into a max flow problem?



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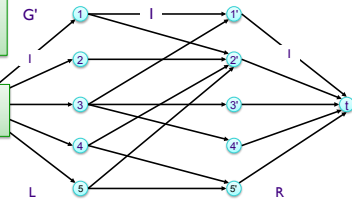
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Review: Bipartite Graph: Max Flow Formulation

- Create digraph $G' = (L \cup R \cup \{s, t\}, E')$
- Direct all edges from L to R, and assign unit capacity
- Add source s, and unit capacity edges from s to each node in L
- Add sink t, and unit capacity edges from each node in R to t

What is cost of generating model?

What is C in this model?



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Summary of Approach

1. Model problem as a flow network
2. Run Ford-Fulkerson algorithm
3. Analyze running time
 - Creating model
 - FF algorithm

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EXTENSIONS TO MAX FLOW

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Circulation with Demands

- Circulation with demands
 - Directed graph $G = (V, E)$
 - Edge capacities $c(e)$, $e \in E$
 - Node supply and demands $d(v)$, $v \in V$
 - $d(v) > 0 \rightarrow$ demand
 - $d(v) < 0 \rightarrow$ supply
 - $d(v) = 0 \rightarrow$ transshipment

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Circulation with Demands

- Circulation with demands
 - Directed graph $G = (V, E)$
 - Edge capacities $c(e)$, $e \in E$
 - Node supply and demands $d(v)$, $v \in V$
 - ↑
demand if $d(v) > 0$; supply if $d(v) < 0$; transshipment if $d(v) = 0$
- Def. A **circulation** is a function that satisfies:
 - For each $e \in E$: $0 \leq f(e) \leq c(e)$ (capacity)
 - For each $v \in V$: $\sum_{e \text{ in to } v} f(e) - \sum_{e \text{ out of } v} f(e) = d(v)$ (conservation)

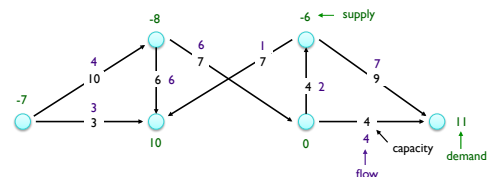
Circulation problem:

given (V, E, c, d) , does a circulation exist?
(Can we satisfy demand with supply?)

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Example Graph: Circulation with Demands



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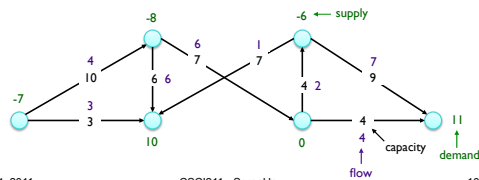
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Circulation with Demands

- Necessary condition:
sum of supplies = sum of demands

$$\sum_{v: d(v) > 0} d(v) = \sum_{v: d(v) < 0} -d(v) =: D$$



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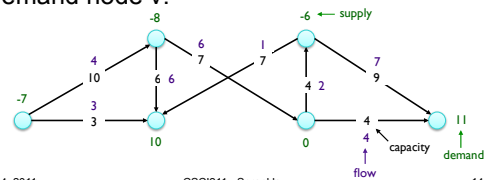
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Circulation with Demands

- Necessary condition:
sum of supplies = sum of demands

$$\sum_{v: d(v) > 0} d(v) = \sum_{v: d(v) < 0} -d(v) =: D$$

- Pf. Sum conservation constraints for every demand node v .



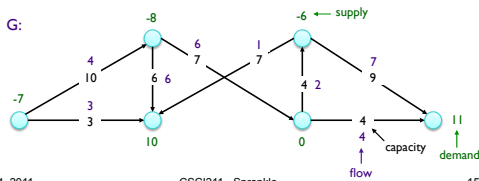
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Circulation with Demands: Towards Max Flow Formulation

Ideas about how we can formulate this as a max flow problem?



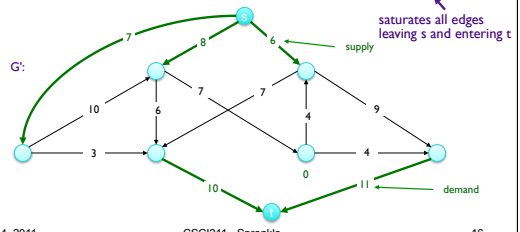
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Circulation with Demands: Max Flow Formulation

- Add new source s and sink t
- For each v with $d(v) < 0$, add edge (s, v) with capacity $-d(v)$
- For each v with $d(v) > 0$, add edge (v, t) with capacity $d(v)$
- Claim: G has circulation iff G' has max flow of value D



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Circulation with Demands: Characterization

- Given (V, E, c, d) , there does **not** exist a circulation iff there exists a node partition (A, B) such that

$$\sum_{v \in B} d_v > \text{cap}(A, B)$$

demand by nodes in B exceeds

supply of nodes in B + max capacity of edges going from $A \rightarrow B$

- Pf?

➤ What can we use to prove this?

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Circulation with Demands: Characterization

- Given (V, E, c, d) , there does **not** exist a circulation iff there exists a node partition (A, B) such that

$$\sum_{v \in B} d_v > \text{cap}(A, B)$$

demand by nodes in B exceeds

supply of nodes in B + max capacity of edges going from $A \rightarrow B$

- Pf idea. Look at min cut in G' .

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ANOTHER EXTENSION: LOWER BOUNDS

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Circulation with Demands and Lower Bounds

- Feasible circulation

- Directed graph $G = (V, E)$
- Edge capacities $c(e)$ and lower bounds $\ell(e)$, $e \in E$
- Node supply and demands $d(v)$, $v \in V$

Force flow to use
certain edges

- Def. A *circulation* is a function that satisfies:

- For each $e \in E$: $0 \leq \ell(e) \leq f(e) \leq c(e)$ (capacity)
- For each $v \in V$: $\sum_{e \text{ in to } v} f(e) - \sum_{e \text{ out of } v} f(e) = d(v)$ (conservation)

Circulation problem with lower bounds.

Given (V, E, ℓ, c, d) , does a circulation exist?

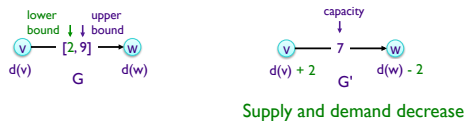
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Circulation with Demands and Lower Bounds

- Model lower bounds with demands
 - Send $\ell(e)$ units of flow along edge e
 - Update demands of both endpoints



Proof in book

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7.8 SURVEY DESIGN

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Survey Design

- Design survey asking consumers about products
- Can only survey a consumer about a product if they own it
 - Consumer can own multiple products
- Ask consumer i between c_i and c_i' questions
- Ask between p_j and p_j' consumers about product j

Goal: Design a survey that meets these specs, if possible.

How can we model this problem?

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Bipartite Graph

- Nodes: customers and products
- Edge between customer and product means customer owns product
- For each customer, range of # of products asked about
- For each product, range of # of customers asked about it

What does the flow represent?

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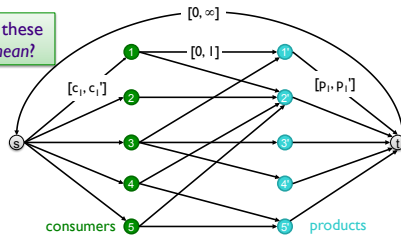
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Survey Design Algorithm

- Formulate as a circulation problem with lower bounds
 - Include an edge (i, j) if customer i owns product j

What do these edges mean?



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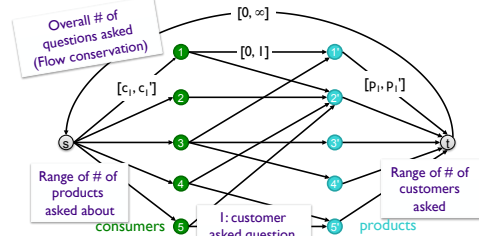
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Survey Design Algorithm

How do we know if we can create a survey?
What is the survey?

- Formulate as a circulation problem with lower bounds
 - Include an edge (i, j) if customer i owns product j



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Survey Solution

- If a feasible, integer solution, can create the survey
- Customer i will be surveyed about product j iff the edge (i, j) carries a unit of flow

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This Week

- Wiki due Wednesday
 - 6.9, 7.1-7.3
- Friday
 - Problem Set 9
 - Read "The status of the P versus NP problem"
- Course evaluations "open" on Wednesday in Sakai
 - Return by following Sunday evening

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