Objectives

- Network Flow
  - Max flow
  - Min cut

Dynamic Programming Wrapup

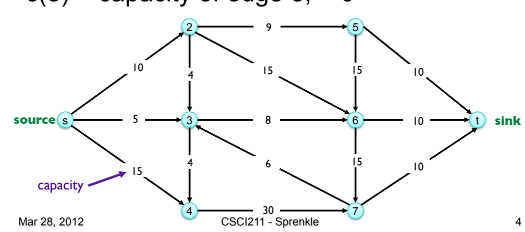
- What we didn't cover
  - 6.5: RNA Secondary Structure: Dynamic Programming Over Intervals
  - 6.9: Shortest Paths and Distance Vector Protocols
    - In practice

Motivating Flow Network Problems

- Modeling *transportation* networks
  - Edges: carry traffic
  - Nodes: pass traffic between edges
- Can represent many different types of problems
  - Instead of looking at all possibilities, formulate as a flow problem

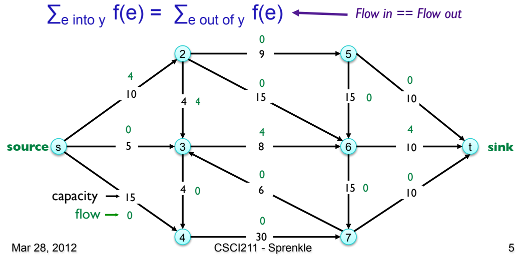
Flow Network

- G = (V, E) = directed graph, no parallel edges
- Two distinguished nodes: *s* = source, *t* = sink
- c(e) = capacity of edge e, > 0



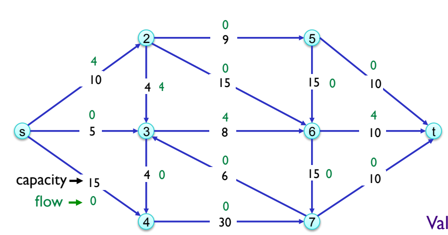
Flows

- An **s-t flow** is a function that satisfies
  - ***Capacity* condition**: For each $e \in E$: $0 \leq f(e) \leq c(e)$ — *Flow can't exceed capacity*
  - ***Conservation* condition**: For each $v \in V - \{s, t\}$:
    $\sum_{e \text{ into } y} f(e) = \sum_{e \text{ out of } y} f(e)$ ← *Flow in == Flow out*



Flows

- The **value** of a flow f is $v(f) = \sum_{e \text{ out of } \mathbf{s}} f(e)$

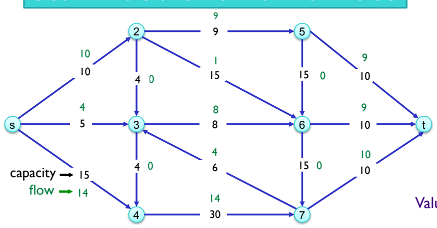


Value = 4

Maximum Flow Problem

- Make network most efficient
  - Use most of available capacity

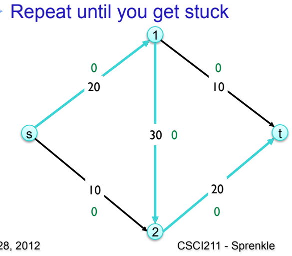
Goal: Find *s-t* flow of maximum value



Value = 28

Towards a Max Flow Algorithm

- Greedy algorithm
  - Start all edges e ∈ E at f(e) = 0
  - Find an *s-t* path P with the most capacity: f(e) < c(e)
  - Augment flow along path P
  - Repeat until you get stuck



Flow value = 0

Towards a Max Flow Algorithm

- Greedy algorithm
  - Start all edges e ∈ E at f(e) = 0
  - Find an *s-t* path P with the most capacity: f(e) < c(e)
  - Augment flow along path P
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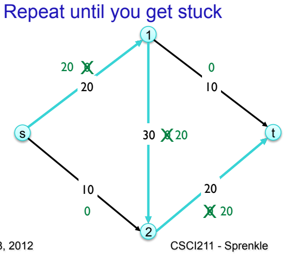


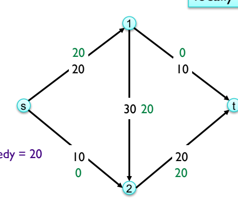
Is this optimal?

Flow value = 20

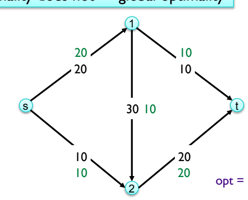
Towards a Max Flow Algorithm

- Greedy algorithm
  - Start all edges e ∈ E at f(e) = 0
  - Find an *s-t* path P with the most capacity: f(e) < c(e)
  - Augment flow along path P
  - Repeat until you get stuck

locally optimality does not ⇒ global optimality



greedy = 20



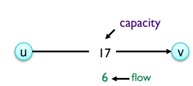
opt = 30

Towards a solution…

RESIDUAL GRAPHS

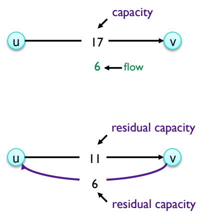
Towards a Residual Graph

- Original edge: e = (u, v) ∈ E
  - Flow f(e), capacity c(e)



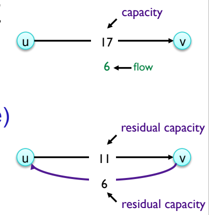
Towards a Residual Graph

- Original edge: e = (u, v) ∈ E
  - Flow f(e), capacity c(e)
- Residual edge
  - e = (u, v) w/ capacity c(e) - f(e)
  - e^R = (v, u) with capacity f(e)
    - To undo flow



Residual Graph: G_f

- Original edge: e = (u, v) ∈ E
  - Flow f(e), capacity c(e)
- Residual edge
  - e = (u, v) w/ capacity c(e) - f(e)
  - e^R = (v, u) with capacity f(e)
    - To undo flow
- Residual graph: $G_f = (V, E_f)$
  - Residual edges with *positive* residual capacity
  - $E_f = \underbrace{\{e : f(e) < c(e)\}}_{\text{Forward edges}} \cup \underbrace{\{e^R : f(e) > 0\}}_{\text{Backward edges}}$



Applying Residual Graph

- Used to find the maximum flow
  - Use similar idea to greedy algorithm
- Residual path: simple *s*-*t* path in G_f
  - Also known as *augmenting path*

Augmenting Path Algorithm

c=capacity

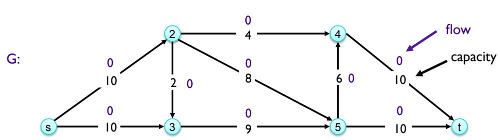
```
Ford-Fulkerson(G, s, t, c)
   foreach e ∈ E  f(e) = 0  # initially no flow
   Gf = residual graph

   while there exists augmenting path P
      f = Augment(f, c, P)     # change the flow
      update Gf               # build a new residual graph

   return f
```

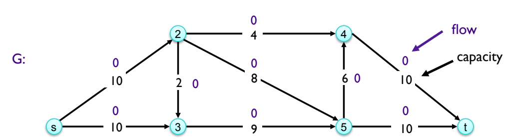
```
Augment(f, c, P)
   b = bottleneck(P) # edge on P with least capacity
   foreach e ∈ P
      if (e ∈ E) f(e) = f(e) + b  # forward edge, ↑ flow
      else       f(eR) = f(e) - b  # forward edge, ↓ flow
   return f
```


Ford-Fulkerson Algorithm



Flow value = 0

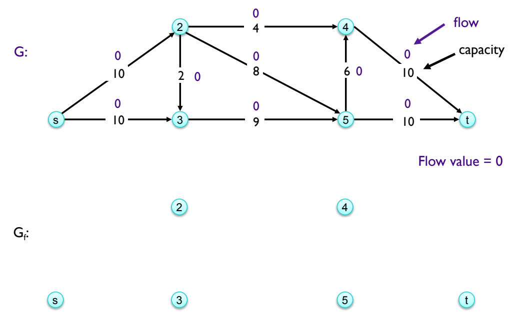
Ford-Fulkerson Algorithm



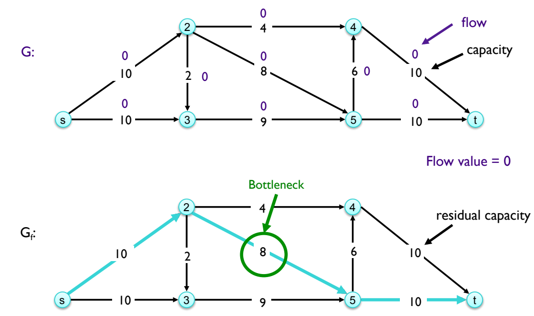
Flow value = 0

What does the residual graph look like?

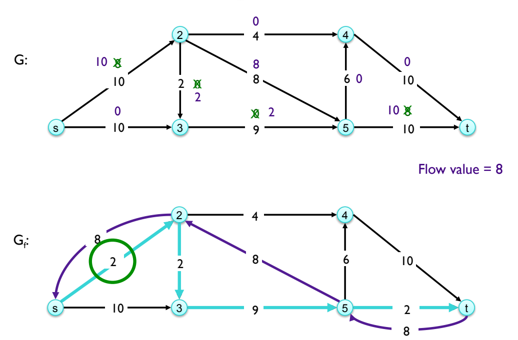
Ford-Fulkerson Algorithm



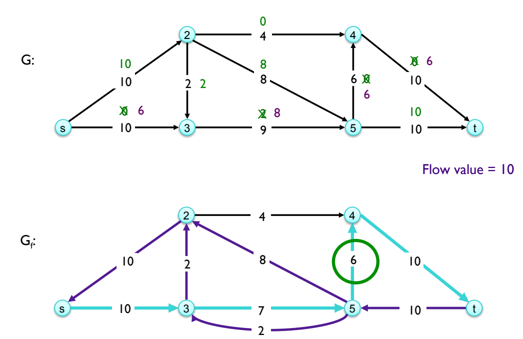
Ford-Fulkerson Algorithm



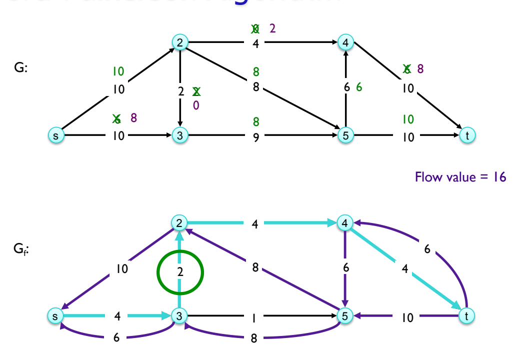
Ford-Fulkerson Algorithm



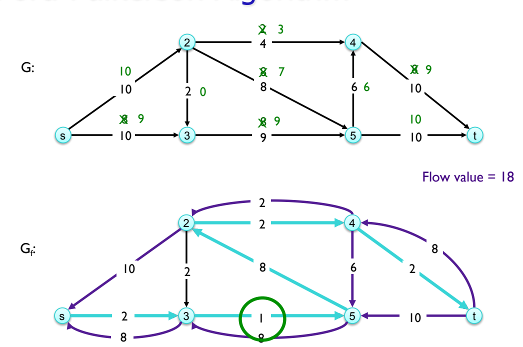
Ford-Fulkerson Algorithm



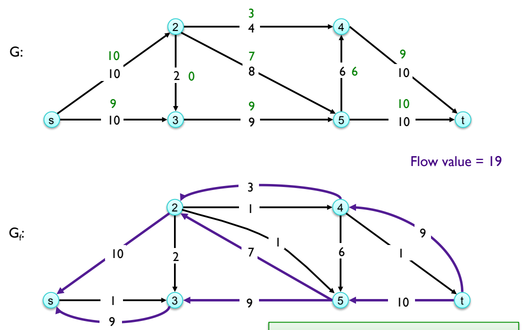
Ford-Fulkerson Algorithm



Ford-Fulkerson Algorithm

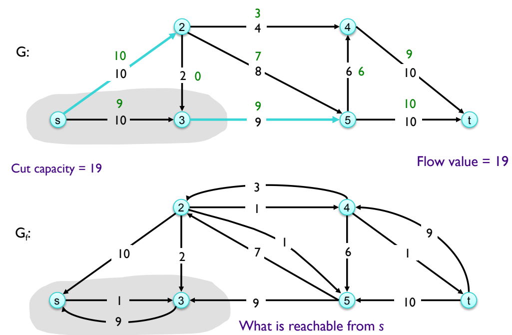


Ford-Fulkerson Algorithm



How do we know we're done?

Ford-Fulkerson Algorithm



Analyzing Augmenting Path Algorithm

```
Ford-Fulkerson(G, s, t, c)
   foreach e ∈ E  f(e) = 0  # initially no flow
   Gf = residual graph

   while there exists augmenting path P
      f = Augment(f, c, P)     # change the flow
      update Gf              # build a new residual graph

   return f
```

```
Augment(f, c, P)
   b = bottleneck(P) # edge on P with least capacity
   foreach e ∈ P
      if (e ∈ E) f(e) = f(e) + b  # forward edge, ↑ flow
      else       f(eR) = f(e) - b  # forward edge, ↓ flow
   return f
```

Why does alg work? What is happening at each iteration?
What is the running time?

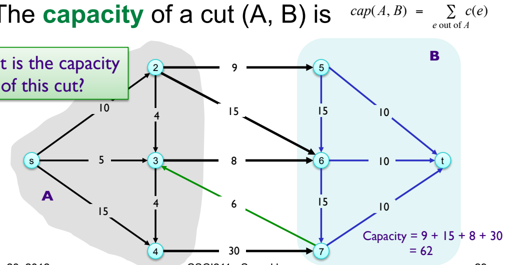
Need more analysis …

MINIMUM CUTS

Cuts

- An **s-t cut** is a partition (A, B) of V with s ∈ A and t ∈ B
- The **capacity** of a cut (A, B) is $cap(A, B) = \sum_{e \text{ out of } A} c(e)$

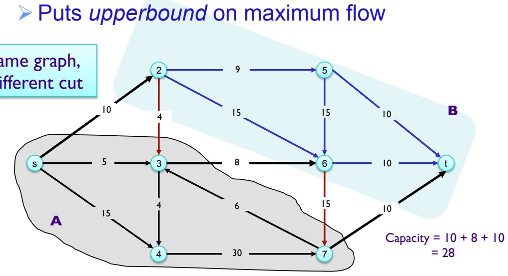
What is the capacity of this cut?



Minimum Cut Problem

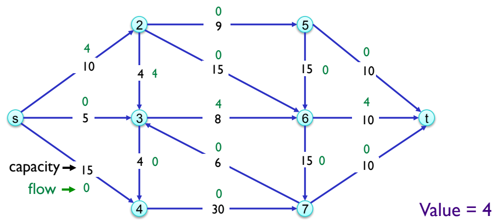
- Find an *s-t* cut of *minimum* capacity
  - Puts *upperbound* on maximum flow

Same graph, different cut



Recall

- The **value** of a flow f is $v(f) = \sum_{e \text{ out of } s} f(e)$

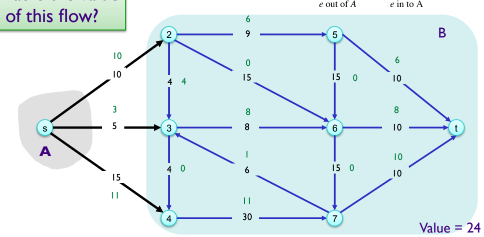


Flow Value Lemma

- Let *f* be *any* flow, and let (A, B) be *any s-t* cut. Then, the value of the flow is = $f^{out}(A) - f^{in}(A)$.

What is the value of this flow?

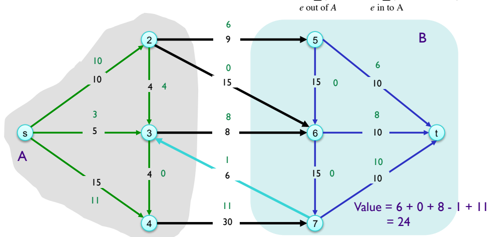
$$\sum_{e \text{ out of } A} f(e) - \sum_{e \text{ in to } A} f(e) = v(f)$$



Flow Value Lemma

- Let *f* be *any* flow, and let (A, B) be *any s-t* cut. Then, the value of the flow is = $f^{out}(A) - f^{in}(A)$.

$$\sum_{e \text{ out of } A} f(e) - \sum_{e \text{ in to } A} f(e) = v(f)$$

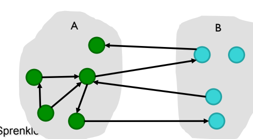


Flow Value Lemma

- Let *f* be *any* flow, and let (A, B) be *any s-t* cut.
- Then $\sum_{e \text{ out of } A} f(e) - \sum_{e \text{ in to } A} f(e) = v(f)$.
- Pf.

$$\begin{aligned} v(f) &= \sum_{e \text{ out of } s} f(e) && \text{By definition} \\ &= \sum_{v \in A} \left(\sum_{e \text{ out of } v} f(e) - \sum_{e \text{ in to v}} f(e) \right) && \text{by flow conservation, all terms except v = s are 0} \\ &= \sum_{e \text{ out of } A} f(e) - \sum_{e \text{ in to A}} f(e). \end{aligned}$$

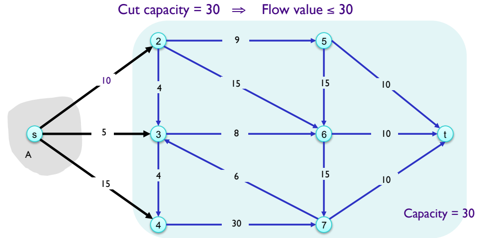
Possibilities for edge e:
- Both ends in A (0)
- Points out from A (+)
- Points in to A (-)



Weak Duality

- Let *f* be any flow and let (A, B) be any *s-t* cut. Then the value of the flow is *at most* the cut's capacity

Cut capacity = 30 ⇒ Flow value ≤ 30

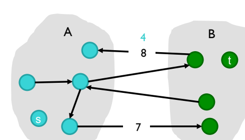


Weak Duality

- Let *f* be any flow. Then, for any *s-t* cut (A, B) v(f) ≤ cap(A, B).
- Pf.

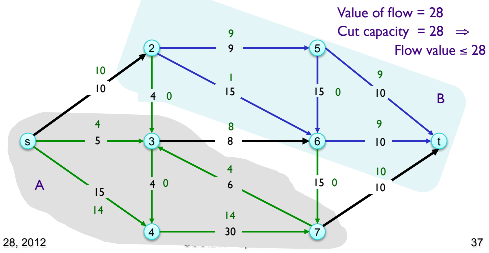
By FVL

$$\begin{aligned} v(f) &= \sum_{e \text{ out of } A} f(e) - \sum_{e \text{ in to } A} f(e) \\ &\le \sum_{e \text{ out of } A} f(e) \\ &\le \sum_{e \text{ out of } A} c(e) \\ &= \text{cap}(A, B) \end{aligned}$$



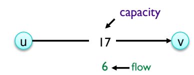
Certificate of Optimality

- Corollary. Let *f* be any flow, and let (A, B) be any cut. If v(f) = cap(A, B), then *f* is a max flow and (A, B) is a min cut.

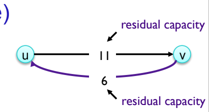


Recall: Residual Graph G_f

- Original edge: e = (u, v) ∈ E
  - Flow f(e), capacity c(e)



- Residual edge
  - e = (u, v) w/ capacity c(e) - f(e)
  - e^R = (v, u) with capacity f(e)
    - To undo flow



- Residual graph: $G_f = (V, E_f)$
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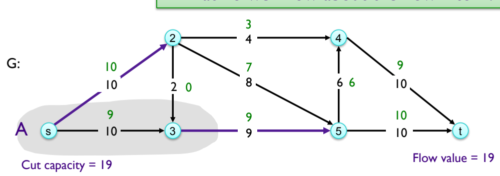
```

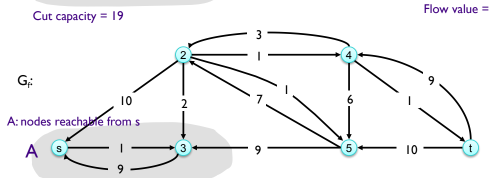

Intuition Behind Correctness of F-F Algorithm

- Let A be set of vertices *reachable* from *s* in residual graph at end of F-F alg execution
- By definition of A, s ∈ A
- By definition of the F-F algorithm's resulting flow, t ∉ A

Ford-Fulkers

- What do we know about the flow out of A?
- What do we know about the flow into A?





This Week

- Problem Set 8 due Friday
- Start reading chapter 7