## Objectives

- Greedy Algorithms
  - Interval partitioning
  - Minimizing Lateness
- Exchange argument



## Review

- What is the template for a greedy solution?
- What problems did we solve optimally with a greedy algorithm?
- How did we prove optimality?



## Review: Greedy Algorithms

- Template
  1. Consider jobs (or whatever) in some order
     - Decision: What order is best?
  2. Take each job provided it's compatible with the ones already taken
- At each step, take as much as you can get
  - Feasible – satisfy problem's constraints
  - Locally optimal – best local choice among available feasible choices
  - Irrevocable – after decided, no going back



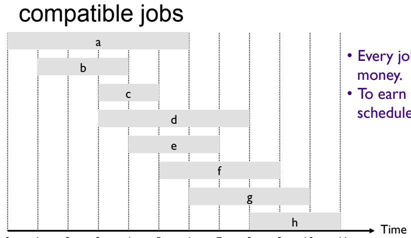
## Review: Greedy Stays Ahead Proofs

1. Define your solutions
   - Describe the form of your greedy solution and of some other solution (possibly the optimal solution)
     - Example: Let A be the solution constructed by the greedy algorithm and O be an solution.
2. Find a measure
   - Find a measure by which greedy stays ahead of the optimal solution
     - Ex: Let $a_1, \ldots, a_k$ be the first k measures of greedy algorithm and $o_1, \ldots, o_m$ be the first m measures of other solution (sometimes $m = k$ )
3. Prove greedy stays ahead
   - Show that the partial solutions constructed by greedy are always just as good as the initial segments of the optimal solution, based on the measure
     - Ex: for all indices $r \le \min(k,m)$, prove by induction that $a_r \ge o_r$ or $a_r \le o_r$
   - Use the greedy algorithm to help you argue the inductive step
4. Prove optimality
   - Prove that since greedy stays ahead of the other solution with respect to the measure, then the greedy solution is optimal.



## Review: Interval Scheduling

- Job j starts at $s_j$ and finishes at $f_j$
- Two jobs are **compatible** if they don't overlap
- **Goal**: find maximum subset of mutually compatible jobs




- Every job is worth equal money.
- To earn the most money → schedule the most jobs



## Problem Assumptions

- All requests were known to scheduling algorithm
  - Online algorithms: make decisions without knowledge of future input
- Each job was worth the same amount
  - What if jobs had *different* values?
    - E.g., scaled with size
- Single resource requested
  - Rejected requests that didn't fit

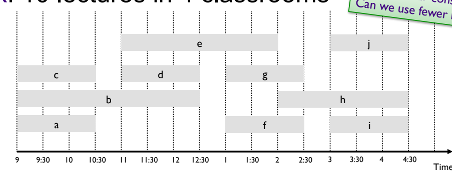
# INTERVAL PARTITIONING



## Interval Partitioning

- Lecture j starts at $s_j$ and finishes at $f_j$
- Goal: find minimum number of classrooms to schedule all lectures so that no two occur at the same time in the same room.
- Ex: 10 lectures in 4 classrooms

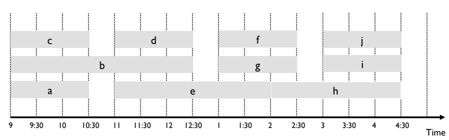
What are our constraints? Can we use fewer rooms?






## Interval Partitioning

- Lecture j starts at $s_j$ and finishes at $f_j$
- Goal: find minimum number of classrooms to schedule all lectures so that no two occur at the same time in the same room.
- Alternative schedule uses only 3 classrooms



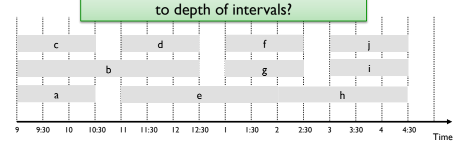



## Interval Partitioning: Lower Bound on Optimal Solution

- Def. The depth of a set of open intervals is the maximum number that contain any given time.
- Key observation. # of classrooms needed ≥ depth.
- Ex: Depth of schedule below = 3 ⇒ schedule below is optimal.

a, b, c all contain 9:30

Does there always exist a schedule equal to depth of intervals?






## Interval Partitioning Discussion

- Does there always exist a schedule equal to depth of intervals?
- Can we make decisions locally to get a global optimum?
  - Or are there long-range obstacles that require more resources?



## Interval Partitioning: Greedy Algorithm

- Consider lectures in increasing order of start time: assign lecture to any compatible classroom

```
Sort intervals by starting time so that s1 ≤ s2 ≤ ... ≤ sn
d = 0   ← number of allocated classrooms
for j = 1 to n
   if lecture j is compatible with some classroom k
      schedule lecture j in classroom k
   else
      allocate a new classroom d + 1
      schedule lecture j in classroom d + 1
      d = d + 1
```

Analyze algorithm

## Interval Partitioning: Greedy Algorithm

- Consider lectures in increasing order of start time: assign lecture to any compatible classroom

```
Sort intervals by starting time so that s1 ≤ s2 ≤ ... ≤ sn
d = 0   <-- number of allocated classrooms
for j = 1 to n
   if (lecture j is compatible with some classroom k)
      schedule lecture j in classroom k
   else
      allocate a new classroom d + 1
      schedule lecture j in classroom d + 1
      d = d + 1
```

- Implementation: O(n log n)
  - For each classroom k, maintain the finish time of the last job added.
  - Keep the classrooms in a priority queue by last job finish time.



## Interval Partitioning: Greedy Analysis

- Observation. Greedy algorithm never schedules two incompatible lectures in the same classroom
- Theorem. Greedy algorithm is optimal
- Pf Intuition
  - When do we add more classrooms?
  - When would we add the d+1 classroom?



## Interval Partitioning: Greedy Analysis

- Observation. Greedy algorithm never schedules two incompatible lectures in the same classroom
- Theorem. Greedy algorithm is optimal
- Pf.
  - Let d = number of classrooms that the greedy algorithm allocates
  - Classroom d is opened because we needed to schedule a job, say j, that is incompatible with all d-1 other classrooms
  - Since we sorted by start time, all these incompatibilities are caused by lectures that start no later than $s_j$
  - Thus, we have d lectures overlapping at time $s_j + \varepsilon$
  - d is the depth of the set of lectures



Exchange argument

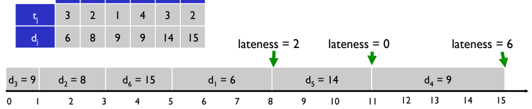
## SCHEDULING TO MINIMIZE MAX LATENESS



## Scheduling to Minimizing Max Lateness

- Single resource processes one job at a time
- Job j requires $t_j$ units of processing time and is due at time $d_j$ (its deadline)
- If j starts at time $s_j$, it finishes at time $f_j = s_j + t_j$
- Lateness: $\ell_j = \max \{ 0, f_j - d_j \}$
- **Goal**: schedule all jobs to *minimize* **maximum lateness** $L = \max \ell_j$

| | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| $t_j$ | 3 | 2 | 1 | 4 | 3 | 2 |
| $d_j$ | 6 | 8 | 9 | 9 | 14 | 15 |




Note: not a sum total



## Greedy Algorithms

- Greedy template. Consider jobs in some order.
- What do we want to optimize?
- What order?
  - Intuition of order?
  - Counter examples for order being optimal?

## Minimizing Lateness: Greedy Algorithms

- Greedy template. Consider jobs in some order.
  - ➢ Shortest processing time first. Consider jobs in ascending order of processing time $t_j$.

Counter example

| | 1 | 2 |
|---|---|---|
| $t_j$ | 1 | 10 |
| $d_j$ | 100 | 10 |

  - ➢ Smallest slack. Consider jobs in ascending order of slack $d_j - t_j$.

Counter example

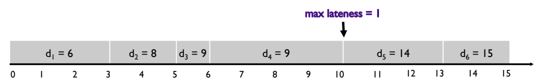
| | 1 | 2 |
|---|---|---|
| $t_j$ | 1 | 10 |
| $d_j$ | 2 | 10 |



## Minimizing Lateness: Greedy Algorithm

- Earliest deadline first.

```
Sort n jobs by deadline so that d1 ≤ d2 ≤ … ≤ dn
t = 0
for j = 1 to n
   Assign job j to interval [t, t + tj]
   sj = t
   fj = t + tj
   t = t + tj
output intervals [sj, fj]
```

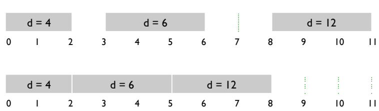



What can we say about this algorithm/its results?



## Minimizing Lateness: No Idle Time

- Observation. There exists an optimal schedule with no idle time




- Observation. The greedy schedule has no idle time



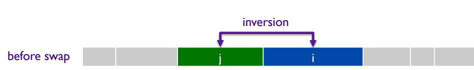
## Proving Optimality

- Goal: Prove greedy algorithm produces optimal solution
- Approach: **Exchange argument**
  - ➢ Start with an optimal schedule Opt
  - ➢ Gradually modify Opt, preserving its optimality
  - ➢ Transform into a schedule identical to greedy's schedule



## Minimizing Lateness: Inversions

- Def. An ***inversion*** in schedule S is a pair of jobs i and j such that:
  $d_i < d_j$ but j scheduled before i

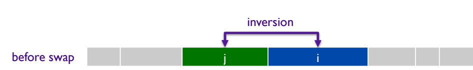



Can Greedy's solution have any inversions?



## Minimizing Lateness: Inversions

- Def. An ***inversion*** in schedule S is a pair of jobs i and j such that:
  $d_i < d_j$ but j scheduled before i




Greedy's schedule has no inversions!

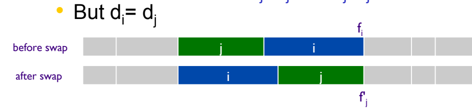
## Minimizing Lateness: Inversions

- Claim. Swapping two adjacent jobs with the same deadline does not increase the max lateness
- Pf Sketch. Let $\ell$ be the lateness before the swap, and let $\ell'$ be it afterwards
  - Lateness of other jobs?
  - Lateness of i? j?




## Minimizing Lateness: Inversions

- Claim. Swapping two adjacent jobs with the same deadline does not increase the max lateness
- Pf. Let $\ell$ be the lateness before the swap, and let $\ell'$ be it afterwards
  - Lateness remains the same for all other jobs:
    - $\ell'_k = \ell_k$ for all $k \neq i, j$
  - Lateness of i before is $f_i - d_i = t_i + t_j - d_i$
  - Lateness of j after is $f_j' - d_j = t_i + t_j - d_j$
    - But $d_i = d_j$





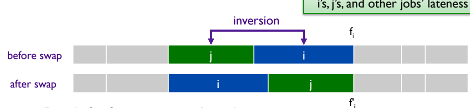

## Minimizing Lateness: Inversions

- Claim. Swapping two adjacent, inverted jobs reduces the number of inversions by one and does *not increase the max lateness*

How do we know inversions are adjacent?

- Pf Setup. Let $\ell$ be the lateness before the swap, and let $\ell'$ be it afterwards

What can we say about how i's, j's, and other jobs' lateness changes?




By def of inversion, $d_i < d_j$



## Minimizing Lateness: Inversions

- Claim. Swapping two adjacent, inverted jobs reduces the number of inversions by one and does *not increase the max lateness*.
- Pf. Let $\ell$ be the lateness before the swap, and let $\ell'$ be it afterwards
  - $\ell'_k = \ell_k$ for all $k \neq i, j$
  - $\ell'_i \leq \ell_i$
  - If job j is late:

$$
\begin{aligned}
\ell'_j &= f'_j - d_j && \text{(definition)} \\
&= f_i - d_j && (j \text{ finishes at time } f_i) \\
&\leq f_i - d_i && (i < j) \\
&\leq \ell_i && \text{(definition)}
\end{aligned}
$$



## Minimizing Lateness: Analysis of Greedy Algorithm

- Theorem. Greedy schedule S is optimal
- Pf idea. Convert Opt to Greedy
  - Does opt schedule have idle time?
  - What if opt schedule has no inversions?
  - What if opt schedule has inversions?



## Minimizing Lateness: Analysis of Greedy Algorithm

- Theorem. Greedy schedule S is optimal
- Pf. Define S* to be an optimal schedule that has the fewest number of inversions, and let's see what happens
  - Can assume S* has no idle time
  - If S* has no inversions, then S = S*
  - If S* has an inversion, let i-j be an adjacent inversion
    - Swapping i and j does not increase the maximum lateness and strictly decreases the number of inversions
    - This contradicts definition of S* ▪

## Greedy Exchange Proofs

1. Label your algorithm's solution and a general solution.
   - Example: let $A = \{a_1, a_2, ..., a_k\}$ be the solution generated by your algorithm, and let $O = \{o_1, o_2, ..., o_m\}$ be an arbitrary (or optimal) feasible solution.
2. Compare greedy with other solution.
   - Assume that your arbitrary/optimal solution is not the same as your greedy solution (since otherwise, you are done).
   - Typically, can isolate a simple example of this difference, such as:
     1. There is an element $e \in O$ that $\notin A$ and an element $f \in A$ that $\notin O$
     2. 2 consecutive elements in O are in a different order than in A (i.e., there is an *inversion*).
3. Exchange.
   - Swap the elements in question in O (either ① swap one element out and another in or ② swap the order of the elements) and argue that solution is no worse than before.
   - Argue that if you continue swapping, you eliminate all differences between O and A in a *finite* # of steps without worsening the solution's quality.
   - Thus, the greedy solution produced is just as good as any optimal solution, and hence is optimal itself.

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## Greedy Analysis Strategies

- Greedy algorithm stays ahead. Show that after each step of the greedy algorithm, its solution is at least as good as any other algorithm's.
- Exchange argument. Gradually transform any solution to the one found by the greedy algorithm without hurting its quality.
- Structural. Discover a simple "structural" bound asserting that every possible solution must have a certain value. Then show that your algorithm always achieves this bound.

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## Assignments

- Exam 1 – due next Monday
  - Open book, open notes, open lecture notes
  - I mention explicitly to analyze your algorithms' running times. I will not do that in the future.
- Wed: work period
  - Ask me questions
  - Office Hours: Wed: 2:30-4, Thurs: 2:30-4:30

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