Objectives

- Wrapping up implementing BFS and DFS
- Graph Application: Bipartite Graphs
- Directed Graphs

Analysis

```
BFS(s):
   Discovered[v] = false, for all v
   Discovered[s] = true
   L[0] = {s}
   layer counter i = 0
   BFS tree T = {}
   while L[i] != {}
       L[i+1] = {}
       For each node u ∈ L[i]
           Consider each edge (u,v) incident to u
           if Discovered[v] == false then
               Discovered[v] = true
               Add edge (u, v) to tree T
               Add v to the list L[i + 1]
       i+=1
```

n

O(n^3)

At most n

At most n-1

At most n-1

Analysis: Tighter Bound

```
BFS(s):
   Discovered[v] = false, for all v
   Discovered[s] = true
   L[0] = {s}
   layer counter i = 0
   BFS tree T = {}
   while L[i] != {}
       L[i+1] = {}
       For each node u ∈ L[i]
           Consider each edge (u,v) incident to u
           if Discovered[v] == false then
               Discovered[v] = true
               Add edge (u, v) to tree T
               Add v to the list L[i + 1]
       i+=1
```

n

O(n^2)

At most n

At most n-1

Because we're going to look at each node at most once

Analysis: Even Tighter Bound

```
BFS(s):
   Discovered[v] = false, for all v
   Discovered[s] = true
   L[0] = {s}
   layer counter i = 0
   BFS tree T = {}
   while L[i] != {}
       L[i+1] = {}
       For each node u ∈ L[i]
           Consider each edge (u,v) incident to u
           if Discovered[v] == false then
               Discovered[v] = true
               Add edge (u, v) to tree T
               Add v to the list L[i + 1]
       i+=1
```

n

At most n

O(deg(u))

$\Sigma_{u \in V} \deg(u) = 2m$

→ O(n+m)

Implementing DFS

- Keep nodes to be processed in a *stack*

```
DFS(s):
   Initialize S to be a stack with one element s
   Explored[v] = false, for all v
   Parent[v] = 0, for all v
   DFS tree T = {}
   while S != {}
       Take a node u from S
       if Explored[u] = false
           Explored[u] = true
           Add edge (u, Parent[u]) to T (if u ≠ s)
           for each edge (u, v) incident to u
               Add v to the stack S
               Parent[v] = u
```


Analyzing DFS

O(n+m)

```
DFS(s):
   Initialize S to be a stack with one element s
   Explored[v] = false, for all v
   Parent[v] = 0, for all v
   DFS tree T = {}
   while S != {}
       Take a node u from S
       if Explored[u] = false
           Explored[u] = true
           Add edge (u, Parent[u]) to T (if u ≠ s)
           for each edge (u, v) incident to u
               Add v to the stack S
               Parent[v] = u
```

deg(u)

Analyzing Finding All Connected Components

- How can we find set of all connected components of graph?

```
R* = set of connected components (a set of sets)
while there is a node that does not belong to R*
    select s not in R*
    R = {s}
    while there is an edge (u,v) where u∈R and v∉R
       add v to R
    Add R to R*
```

But the inner loop is O(m+n)! How can this RT be possible?

Running time: O(m+n)

Set of All Connected Components

- How can we find set of all connected components of graph?

```
R* = set of connected components (a set of sets)
while there is a node that does not belong to R*
    select s not in R*
    R = {s}
    while there is an edge (u,v) where u∈R and v∉R
       add v to R
    Add R to R*
```

Imprecision in the running time of inner loop: O(m+n)

But that's m and n of the *connected* component, let's say m_i and n_i.
Σ_i O(m_i+ n_i) = O(m+n)

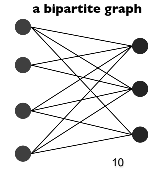
Where i is the subscript of the connected component

BIPARTITE GRAPHS

Bipartite Graphs

- Def. An undirected graph G = (V, E) is ***bipartite*** if the nodes can be colored red or blue such that every edge has one red and one blue end
  - Generally: vertices divided into sets X and Y
- Applications:
  - Stable marriage:
    - men = red, women = blue
  - Scheduling:
    - machines = red, jobs = blue

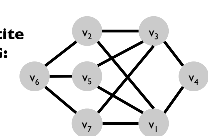
a bipartite graph



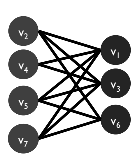
Testing Bipartiteness

- Given a graph G, is it bipartite?
- Many graph problems become:
  - Easier if underlying graph is bipartite (e.g., matching)
  - Tractable if underlying graph is bipartite (e.g., independent set)
- Before designing an algorithm, need to understand structure of bipartite graphs

a bipartite graph G:



another drawing of G:

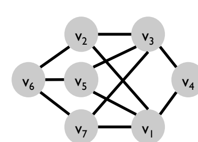


How Can We Determine if a Graph is Bipartite?

- Given a connected graph

Why connected?

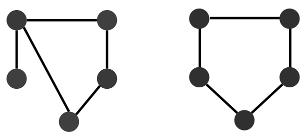
  1. Color one node red
     - Doesn't matter which color (Why?)
  - What should we do next?



- How will we know when we're finished?
- What does this process sound like?

An Obstruction to Bipartiteness

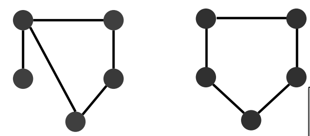
- Lemma. If a graph G is bipartite, it cannot contain an odd-length cycle.



bipartite (2-colorable) **not bipartite (not 2-colorable)**

An Obstruction to Bipartiteness

- Lemma. If a graph G is bipartite, it cannot contain an odd-length cycle.
- Pf. Not possible to 2-color the odd cycle, let alone G.

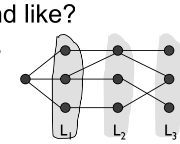


If find an odd cycle, graph is NOT bipartite

bipartite (2-colorable) **not bipartite (not 2-colorable)**

How Can We Determine if a Graph is Bipartite?

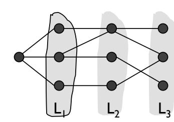
- Given a connected graph
  - Color one node red
    - Doesn't matter which color (Why?)
  - What should we do next?
- How will we know that we're finished?
- What does this process sound like?
  - BFS: alternating colors, layers



How can we implement the algorithm?

Implementing Algorithm

- Modify BFS to have a Color array
- When add v to list L[i+1]
  - Color[v] = red if i+1 is even
  - Color[v] = blue if i+1 is odd



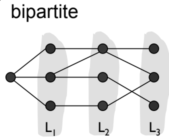
What is the running time of this algorithm? **O(n+m)**

Marks a change in how we think about algorithms
Starting to apply known algorithms to solve new problems

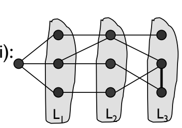
Analyzing Algorithm's Correctness

- Lemma. Let G be a connected graph, and let $L_0, \ldots, L_k$ be the layers produced by BFS starting at node *s*. Exactly one of the following holds:
  - (i) No edge of G joins two nodes of the same layer
    $\Rightarrow$ G is bipartite
  - (ii) An edge of G joins two nodes of the same layer
    $\Rightarrow$ G contains an odd-length cycle and hence is not bipartite

Case (i):

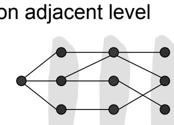


Case (ii):



Analyzing Algorithm's Correctness

- Lemma. Let G be a connected graph, and let $L_0, \ldots, L_k$ be the layers produced by BFS starting at node *s*. Exactly one of the following holds:
  - (i) No edge of G joins two nodes of the same layer
    $\Rightarrow$ G is bipartite
- Pf. (i)
  - Suppose no edge joins two nodes in the same layer
  - Implies all edges join nodes on adjacent level
  - Bipartition
    - red = nodes on odd levels
    - blue = nodes on even levels



Analyzing Algorithm's Correctness

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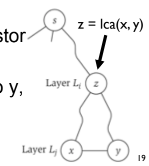
  - (ii) An edge of G joins two nodes of the same layer → G contains an odd-length cycle and hence is not bipartite
- Pf. (ii)
  - Suppose (x, y) is an edge with x, y in same level L_j.
  - Let z = lca(x, y) = lowest common ancestor
  - Let L_i be level containing z
  - Consider cycle that takes edge from x to y, then path y →z, then path from z → x



Analyzing Algorithm's Correctness

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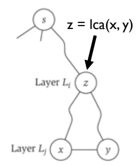
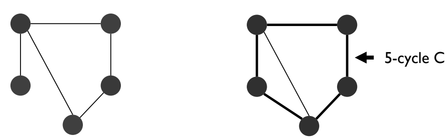
  - Let z = lca(x, y)=lowest common ancestor
  - Let L_i be level containing z
  - Consider cycle that takes edge from x to y, then path y → z, then path z → x
  - Its length is $\underbrace{1}_{(x,y)} + \underbrace{(j-i)}_{\text{path from y to z}} + \underbrace{(j-i)}_{\text{path from z to x}}$, which is odd



An Obstruction to Bipartiteness

- Corollary. A graph G is bipartite *iff* it contains no odd length cycle.



bipartite (2-colorable) **not bipartite (not 2-colorable)**

Looking ahead

- Monday: Andrew Danner
  - 11:15: Public talk
    - Answers to questions on Sakai (10 points)
  - 4:10: external memory algorithms
- Reading Chapter 3.1-3.4
  - Wikis for Tuesday
- For next Friday: Problem Set 3