Objectives

- Network Flow
  - Extensions: Circulation
  - Application: Survey Design
  - Application: Airline Scheduling

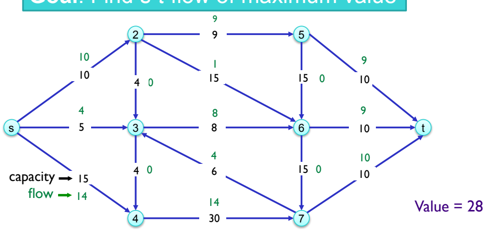
Review

- What are the characteristics of the network flow graph we're dealing with?
- What was the problem we were trying to solve?
- Describe our algorithm to solve the problem
- What is the power of the max-flow/min-cut algorithm?

Maximum Flow Problem

- Make network most efficient
  - Use most of available capacity

Goal: Find *s-t* flow of maximum value



Power of Max Flow Problem

Some problems with non-trivial combinatorial searches can be formulated as **max flow** or **min cut** in a directed graph

Network Flow Solutions

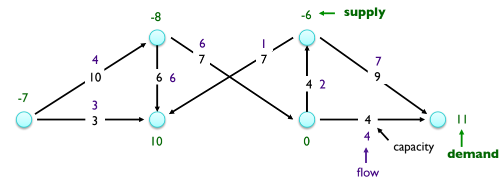
1. Model problem as a flow network
   - Describe what nodes, edges, and capacity represent
   - Describe what flow represents and how that maps to your solution
   - Run Ford-Fulkerson algorithm
2. Prove that the solution found is correct/ feasible/optimal
3. Prove that you find all solutions
4. Analyze running time
   - Creating model
   - FF algorithm

EXTENSIONS TO MAX FLOW

Circulation with Demands

- Directed graph G = (V, E)
- Edge capacities c(e), e ∈ E
- Node supply and demands d(v), v ∈ V
  - d(v) > 0 → demand
  - d(v) < 0 → supply
  - d(v) = 0 → transshipment

Example Graph: Circulation with Demands



- d(v) > 0 → demand
- d(v) < 0 → supply
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Circulation with Demands

- Circulation with demands
  - Directed graph G = (V, E)
  - Edge capacities c(e), e ∈ E
  - Node supply and demands d(v), v ∈ V
- Def. A ***circulation*** is a function that satisfies:
  - For each e ∈ E: $0 \le f(e) \le c(e)$ (capacity)
  - For each v ∈ V: $\sum_{e \text{ in to } v} f(e) - \sum_{e \text{ out of } v} f(e) = d(v)$ (conservation)

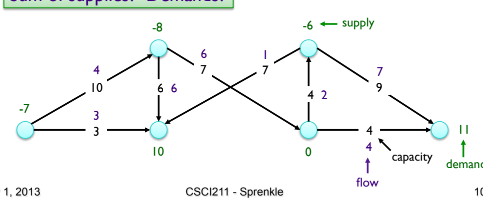
Circulation problem:
given (V, E, c, d), does a circulation exist?
(Can we satisfy demand with supply?)

Circulation with Demands

- Necessary condition:
  sum of supplies = sum of demands

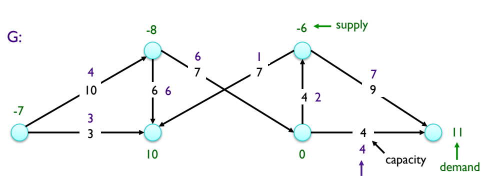
$$\sum_{v\,:\,d(v)>0} d(v) = \sum_{v\,:\,d(v)<0} -d(v) =: D$$

Sum of supplies? Demands?



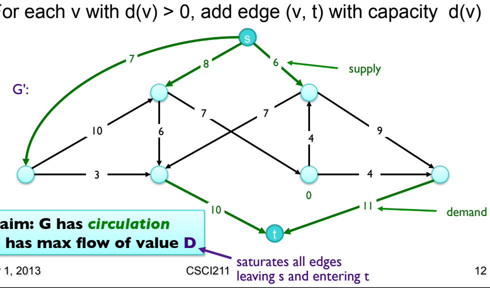
Circulation with Demands: Towards Max Flow Formulation

Ideas about how we can formulate this as a max flow problem?



Circulation with Demands: Max Flow Formulation

- Add source s and sink t
- For each v with d(v) < 0, add edge (s, v) with capacity -d(v)
- For each v with d(v) > 0, add edge (v, t) with capacity d(v)



Claim: G has *circulation* iff G' has max flow of value D

saturates all edges leaving s and entering t

Circulation with Demands: Characterization

- Given (V, E, c, d), there does **not** exist a circulation iff there exists a node partition (A, B) such that

$$\Sigma_{v \in B}\, d_v > \text{cap}(A, B)$$

demand by nodes in B — exceeds — supply of nodes in B + max capacity of edges going from A → B

- Pf?
  - What can we use to prove this?

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Circulation with Demands: Characterization

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demand by nodes in B — exceeds — supply of nodes in B + max capacity of edges going from A → B

- Pf idea. Look at min cut in G'.

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ANOTHER EXTENSION: LOWER BOUNDS

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Circulation with Demands and Lower Bounds

Force flow to use certain edges

- Feasible circulation
  - Directed graph G = (V, E)
  - Edge capacities c(e) and lower bounds ℓ (e), e ∈ E
  - Node supply and demands d(v), v ∈ V
- Def. A *circulation* is a function that satisfies:
  - For each e ∈ E: $0 \le \ell(e) \le f(e) \le c(e)$ (capacity)
  - For each v ∈ V: $\sum_{e \text{ in to } v} f(e) - \sum_{e \text{ out of } v} f(e) = d(v)$ (conservation)

Circulation problem with lower bounds.
Given (V, E, ℓ, c, d), does a circulation exist?

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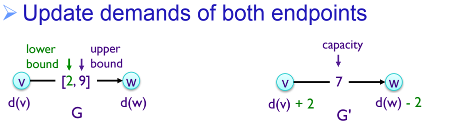
Circulation with Demands and Lower Bounds

- Model lower bounds with demands
  - Send ℓ(e) units of flow along edge e
  - Update demands of both endpoints



Supply and demand decrease

Proof in book

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7.8 SURVEY DESIGN

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Survey Design

- Design survey asking consumers about products
- Can only survey a consumer about a product if they own it
  - Consumer can own multiple products
- Ask consumer i between c_i and c_i' questions
- Ask between p_j and p_j' consumers about product j

Goal: Design a survey that meets these specs, if possible.

How can we model this problem?

Bipartite Graph

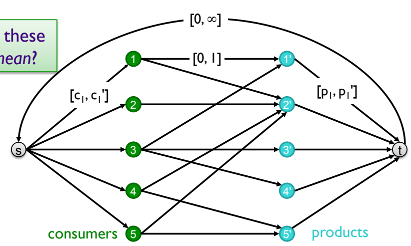
- Nodes: customers and products
- Edge between customer and product means customer owns product
- For each customer, range of # of products asked about
- For each product, range of # of customers asked about it

What does the flow represent?

Survey Design Algorithm

- Formulate as a circulation problem with lower bounds
  - Include an edge (i, j) if customer i owns product j

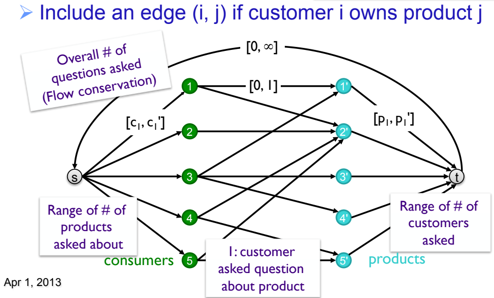
What do these edges *mean?*



Survey Design Algorithm

How do we know if we can create a survey? What is the survey? Does this find all possible solutions?

- Formulate as a circulation problem with lower bounds
  - Include an edge (i, j) if customer i owns product j



Survey Solution

- If a feasible, integer solution, can create the survey
- Customer *i* will be surveyed about product *j* iff the edge (*i,j*) carries a unit of flow
- Many possible survey solutions
  - Model will represent all valid solutions
- Runtime?

This Week

- Wiki Reading
  - 7.1-7.2, 7.5, 7.7
- Problem Set 9 due Friday