## Objectives

- Dynamic Programming
  - Segmented Least Squares
  - Subset Sums



## Review: Weighted Interval Scheduling

1. Determine optimal substructure of problem
   - Define the recurrence relation
2. Define algorithm to find the ***value*** of optimal solution
3. Optionally, change algorithm to an ***iterative*** rather than recursive solution
4. Define algorithm to find ***optimal solution***
5. Analyze running time of algorithms

Map to weighted-interval scheduling



## Review: Iterative Weighted Interval Scheduling Solution

- Build up solution from subproblems instead of breaking down

```
Input: n, s1,…,sn , f1,…,fn , v1,…,vn

Sort jobs by finish times so that f1 ≤ f2 ≤ ... ≤ fn.

Compute p(1), p(2), …, p(n)

M[0] = 0
for j = 1 to n
    M[j] = max(vj + M[p(j)], M[j-1])      O(n)
```

- Typically, we'll take iterative approach



## Review: Weighted Interval Scheduling: Finding a Solution

- Dynamic programming algorithms compute ***optimal value***
- What if we want the ***solution*** itself (**not** simply the value)?
- Do some post-processing

```
M-Compute-Opt(n)
Find-Solution(n)

def Find-Solution(j):
   if j = 0:
      output nothing
   elif vj + M[p(j)] > M[j-1]:
      print j
      Find-Solution(p(j))
   else:
      Find-Solution(j-1)
```

Runtime: O(n)



## SEGMENTED LEAST SQUARES

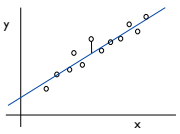


## Least Squares

- Foundational problem in statistic and numerical analysis
- Given $n$ points in the plane: $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$
- Find a line $y = ax + b$ that minimizes the sum of the squared error
  - "line of best fit"

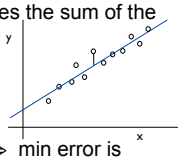
Sum of squared error

$$SSE = \sum_{i=1}^{n} (y_i - ax_i - b)^2$$



## Least Squares

- Foundational problem in statistic and numerical analysis
- Given *n* points in the plane: $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$
- Find a line $y = ax + b$ that minimizes the sum of the squared error
  - "line of best fit"



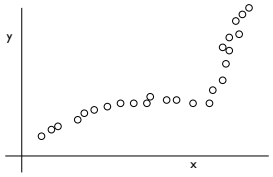

Sum of squared error $SSE = \sum_{i=1}^{n} (y_i - ax_i - b)^2$

- Closed form solution. Calculus ⇒ min error is achieved when $a = \frac{n\sum_i x_i y_i - (\sum_i x_i)(\sum_i y_i)}{n\sum_i x_i^2 - (\sum_i x_i)^2}$, $b = \frac{\sum_i y_i - a\sum_i x_i}{n}$



## Least Squares

- What happens to the error if we try to fit one line to these points?

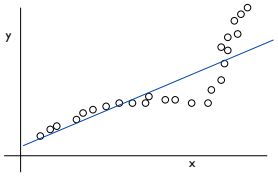



- What pattern does it seem like these points have?



## Least Squares

- What happens to the error if we try to fit one line to these points?
  - Large error



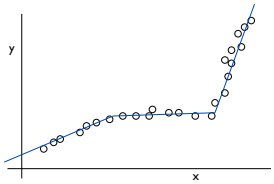

- Pattern: More like 3 lines



## Our Problem: Segmented Least Squares

- Points lie roughly on a **sequence** of line segments
- Given *n* points in the plane $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ with $x_1 < x_2 < \ldots < x_n$, find a **sequence** of **line segments** that **minimizes f(x)**

If I want the ***best*** fit, how many lines should I use?



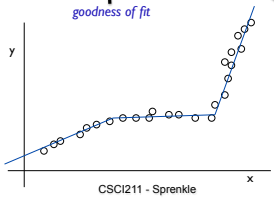



## Segmented Least Squares

- Points lie roughly on a **sequence** of line segments
- Given *n* points in the plane $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ with $x_1 < x_2 < \ldots < x_n$, find a sequence of line segments that **minimizes f(x)**

What's a reasonable choice for *f(x)* to balance *accuracy* and *parsimony*?

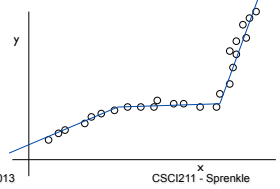
*goodness of fit* *number of lines*






## Segmented Least Squares

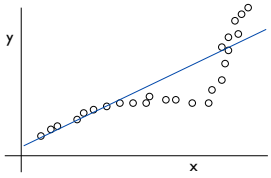
- Points lie roughly on a **sequence** of several line segments.
- Given *n* points in the plane $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ with $x_1 < x_2 < \ldots < x_n$, find a sequence of line segments that minimizes:
  - *E*: sum of the sums of the squared errors in each segment
  - *L*: the number of lines
- **Tradeoff function**: $E + c\,L$, for some constant $c > 0$.




How should we define an optimal solution?

## Segmented Least Squares

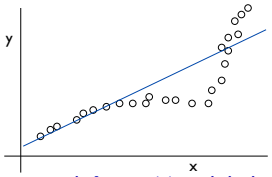
- What made it seem like the points were in 3 lines? What happened?






## Segmented Least Squares

- What made it seem like the points were in 3 lines? What happened?




  - Error increased → want to minimize error
  - Looking for *change* in linear approximation
    - Where to partition points into line segments



## Recall: Properties of Problems for DP

- Polynomial number of subproblems
- Solution to original problem can be easily computed from solutions to subproblems
- Natural ordering of subproblems, easy to compute recurrence

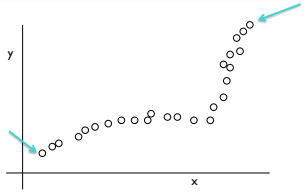
We need to:
- Figure out how to break the problem into subproblems
- Figure out how to compute solution from subproblems
- Define the recurrence relation between the problems



## Toward a Solution

- Consider just the first or last point

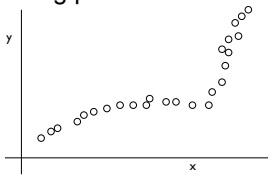
What do we know about those points? their segments? cost of a segment?






## Toward a Solution

- $p_n$ can only belong to one segment
  - Segment: $p_i, \ldots, p_n$
  - Cost: c (cost for segment) + error of segment
- What is the remaining problem?

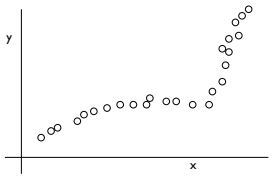





## Toward a Solution

- $p_n$ can only belong to one segment
  - Segment: $p_i, \ldots, p_n$
  - Cost: c (cost for segment) + error of segment
- What is the remaining problem?
  - Solve for $p_1, \ldots, p_{i-1}$

**Next**: Formulate as a recurrence



## Dynamic Programming: Multiway Choice

- Notation.
  - **OPT(j)** = minimum cost for points $p_1$, $p_{i+1}$ , … , $p_j$.
  - **e(i, j)** = minimum sum of squares for points $p_i$, $p_{i+1}$ , …, $p_j$.
- How do we compute OPT(j)?
  - Last problem: binary decision (include job or not)
  - This time: **multiway** decision
    - Which option do we choose?



## Dynamic Programming: Multiway Choice

- Notation.
  - **OPT(j)** = minimum cost for points $p_1$, $p_{i+1}$ , … , $p_j$.
  - **e(i, j)** = minimum sum of squares for points $p_i$, $p_{i+1}$ , …, $p_j$.
- To compute OPT(j):
  - Last segment contains points $p_i$, $p_{i+1}$, … , $p_j$ for some i
  - Cost = e(i, j) + c + OPT(i-1).

$$OPT(j) = \begin{cases} 0 & \text{if } j = 0 \\ \min_{1 \le i \le j} \{ e(i,j) + c + OPT(i-1) \} & \text{otherwise} \end{cases}$$



## Segmented Least Squares: Algorithm

```
INPUT: n, p1,…,pN , c

Segmented-Least-Squares()
   M[0] = 0
   e[0][0] = 0  # needed?
   for j = 1 to n
      for i = 1 to j
         e[i][j] = least square error for the
                   segment pi, …, pj

   for j = 1 to n
      M[j] = min 1 ≤ i ≤ j (e[i][j] + c + M[i-1])

   return M[n]
```

Costs?



## Segmented Least Squares: Algorithm Analysis

How do we find the *solution*?

```
INPUT: n, p1,…,pN , c

Segmented-Least-Squares()
   M[0] = 0
   e[0][0] = 0
   for j = 1 to n
      for i = 1 to j
         e[i][j] = least square error for the
                   segment pi,…, pj

   for j = 1 to n
      M[j] = min 1 ≤ i ≤ j (e[i][j] + c + M[i-1])

   return M[n]
```

can be improved to $O(n^2)$ by pre-computing various statistics

$O(n^3)$

$O(n^2)$

- Bottleneck: computing e(i, j) for $O(n^2)$ pairs, O(n) per pair using previous formula



## Post-Processing: Finding the Solution

```
FindSegments(j):
   if j = 0:
      output nothing
   else:
      Find an i that minimizes ei,j + c + M[i-1]
      Output the segment {pi, …, pj}
      FindSegments(i-1)
```

Cost? $O(n^2)$



## SUBSET SUMS and KNAPSACKS

## The Price is Right

*Or, shopping with someone else's money*

- **Goal**: Spend as much money as possible without going over $100
  - CD $18
  - Jeans $40
  - DVD $35
  - Dinner $15
  - Book $8
  - Ice cream $5
  - Shoes $62
  - Pizza $7

Possible solutions?



## Knapsack Problem

- Given *n* objects and a "knapsack"
- Item *i* weighs $w_i > 0$ kilograms and has value $v_i > 0$
  - Alternative: jobs require $w_i$ time
- Knapsack has capacity of *W* kilograms
  - Alternative: *W* is time interval that resource is available

**Goal**: fill knapsack so as to maximize total **value**

W = 11

| Item | Value | Weight |
|---|---|---|
| 1 | 1 | 1 |
| 2 | 6 | 2 |
| 3 | 18 | 5 |
| 4 | 22 | 6 |
| 5 | 28 | 7 |



## Towards a Recurrence…

- What do we know about the knapsack with respect to item *i*?



## Towards a Recurrence…

- What do we know about the knapsack with respect to item *i*?
  - Either select item *i* or not
  - If don't select
    - Pick optimum solution of remaining items
  - Otherwise
    - What happens?
    - How does problem change?



## Dynamic Programming: False Start

- Def. OPT(i) = max profit subset of items 1, …, i
  - Case 1: OPT does not select item i
    - OPT selects best of { 1, 2, …, i-1 }
  - Case 2: OPT selects item i
    - Accepting item *i* does not immediately imply that we will have to reject other items
      - No known conflicts
    - Without knowing what other items were selected before *i*, we don't even know if we have enough room for *i*

➡Need more sub-problems!



## Dynamic Programming: Adding a New Variable

- Def. OPT(i, w) = max profit subset of items 1, …, i with weight limit w
  - Case 1: OPT does not select item *i*
    - OPT selects best of { 1, 2, …, i-1 } using weight limit *w*
  - Case 2: OPT selects item *i*
    - new weight limit = $w - w_i$
    - OPT selects best of { 1, 2, …, i–1 } using new weight limit

$$OPT(i,w) = \begin{cases} 0 & \text{if } i = 0 \\ OPT(i-1,w) & \text{if } w_i > w \\ \max\{OPT(i-1,w),\ v_i + OPT(i-1,w-w_i)\} & \text{otherwise} \end{cases}$$

## Looking Ahead

- Exam 2 due Friday
- Wednesday work period