Objectives

- Analyzing proofs
- Introduction to problem solving
  - Our process, through an example

Wiki:
- Everyone log in okay?
- Decide on either using a blog or wiki-style journal?

Review

- What are our goals in solving problems?
- How do we show that our solutions are correct and efficient?
- What proof techniques did we discuss?

Proof Summary

- Need to ***prove*** conjectures
- Common types of proofs
  - Direct proofs
  - Contradiction
  - Induction
- Common error: not checking/proving assumptions
  - "Jumps" in logic

Proof: All Horses Are The Same Color

- **Base case**: If there is only *one* horse, there is only one color.
- **Induction step**: Assume as induction hypothesis that within any set of n horses, there is only one color.
  - Look at any set of $n + 1$ horses
  - Label the horses: 1, 2, 3, ..., n, $n + 1$
  - Consider the sets {1, 2, 3, ..., n} and {2, 3, 4, ..., $n + 1$}
  - Each is a set of only n horses, therefore within each there is only one color
  - Since the two sets overlap, there must be only one color among all $n + 1$ horses

Where is the error in the proof?

Error in Proof

- **Base case**: If there is only *one* horse, there is only one color.
- **Induction step**: Assume as induction hypothesis that within any set of *n* horses, there is only one color.
  - Look at any set of *n* + 1 horses
  - Number them: 1, 2, 3, ..., *n*, *n* + 1
  - Consider the sets {1, 2, 3, ..., *n*} and {2, 3, 4, ..., *n* + 1}
  - Each is a set of only *n* horses, therefore within each there is only one color
  - *Since the two sets overlap*, there must be only one color among all *n* + 1 horses

Does not hold true when n+1=2

Lesson: check assumptions within proof

Process, through example

INTRODUCTION TO PROBLEM SOLVING

Matching Residents to Hospitals

- **Goal**: Given a set of preferences among hospitals and medical school students, design a *self-reinforcing* admissions process.

- Applicant *a* and hospital *h* are ***unstable*** if
  - *a* prefers *h* to its assigned hospital
  - *h* prefers *a* to one of its admitted students

- **Stable assignment**: Assignment with no unstable pairs
  - No incentive for some pair of participants to undermine assignment by joint action
    - Unstable pair could each improve their situation by swapping with current assignment
    - Self-reinforcing

What details make this problem tricky?
What info do we need to solve problem?

Stable Matching Problem

Simplified version of resident-matching problem

- **Goal**: Given *n* men and *n* women, find a "suitable" matching
  - Participants rank members of opposite sex
  - Each man ranks women in order of preference
  - Each woman ranks men in order of preference

favorite → 1st; least favorite → 3rd

	1st	2nd	3rd
Xavier	Amy	Bertha	Clare
Yancey	Bertha	Amy	Clare
Zeus	Amy	Bertha	Clare

Men's Preference Profile

	1st	2nd	3rd
Amy	Yancey	Xavier	Zeus
Bertha	Xavier	Yancey	Zeus
Clare	Xavier	Yancey	Zeus

Women's Preference Profile

Stable Matching Goals

- **Perfect matching**: everyone is matched monogamously
  - Each man is paired with exactly one woman
  - Each woman is paired with exactly one man
- **Stability**: no incentive for some pair of participants to undermine assignment by joint action
  - An *unmatched* pair *m*-*w* is **unstable** if man *m* and woman *w* prefer each other to current partners
  - Unstable pair *m*-*w* could each improve by eloping
- **Stable matching**: perfect matching with no unstable pairs

Stable matching problem:
Given the preference lists of *n* men and *n* women, find a stable matching if one exists.

Analyzing Stability

Instable: *m* prefers *w* to his woman; *w* prefers *m* to her man

- Is pairing X-C, Y-B, Z-A stable?

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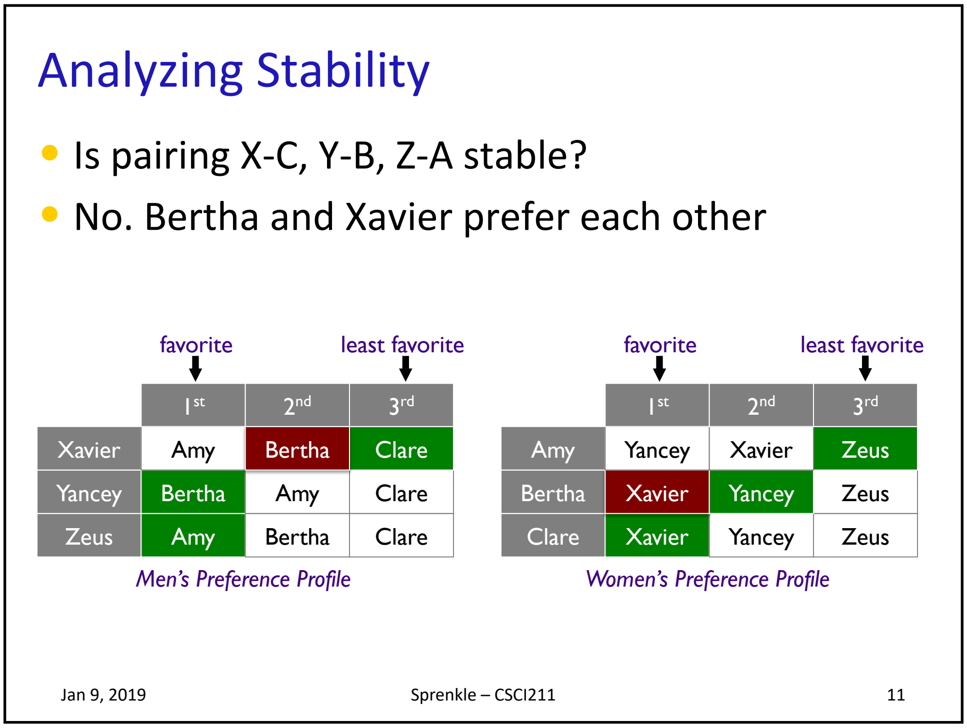

Analyzing Stability

- Is pairing X-C, Y-B, Z-A stable?
- No. Bertha and Xavier prefer each other

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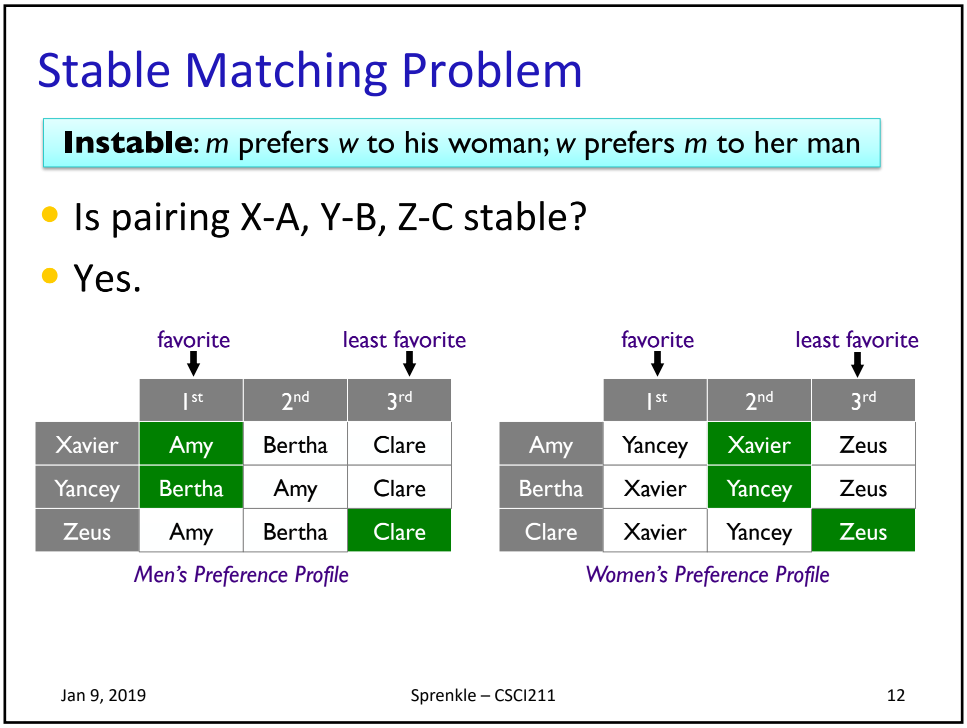

Stable Matching Problem

Instable: *m* prefers *w* to his woman; *w* prefers *m* to her man

- Is pairing X-A, Y-B, Z-C stable?
- Yes.

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Any Questions?

- What are you wondering about this problem/its solution at this point?

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- What are you wondering about this problem/its solution at this point?

- Hopefully:
  - Is there a stable matching for every pair of preference lists?
  - If so, is there an algorithm to find the stable matching?
  - Can we be fair in the matching? (preferences)
  - Will the matching always be the same?

Thoughts on Solving Problem

- What do we need to solve the problem?
- What do we know?
- Where should the state start?
- What are some initial ideas about approaches?

Thoughts on Solving Problem

- Initially, no one is matched
- Pick an arbitrary man and have him match with his favorite woman.
    - Are we guaranteed that pair will be part of a stable matching?
- Should a woman accept her first offer? If not, what should she do?
- When are we done? Do we need to consider all combinations?

Propose-And-Reject Algorithm

[Gale-Shapley 1962]

- Intuitive method that guarantees finding a stable matching

```
Initialize each person to be free
while some man is free and hasn't proposed to every woman
    Choose such a man m
    w = 1st woman on m's list to whom m has not yet proposed
    if w is free
        assign m and w to be engaged
    else if w prefers m to her fiancé m'
        assign m and w to be engaged and m' to be free
    else
        w rejects m
```

Applying the Algorithm

favorite ↓ least favorite ↓

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Assignments

- Review Chapter 1
- Journal due Monday
    - Preface, Chapter 1.1
    - Check out the content requirements for the journal entries