Objectives

- Review implementation of Stable Matching
- Survey of common running times

Turn in completed problem sets

Review: Asymptotic Analysis of Gale-Shapley Alg

Not explicitly in the algorithm, but we need to make the inverse array before the while loop too.

```
Initialize each person to be free     O(n)
while (some man is free and hasn't proposed to every woman)     O(n²)
    Choose such a man m     O(1)
    w = 1st woman on m's list to whom m has not yet proposed     O(1)
    if (w is free)     O(1)
        assign m and w to be engaged     O(1)
    else if (w prefers m to her fiancé m')     O(1)  Using inverse array
        assign m and w to be engaged and m' to be free     O(1)
    else
        w rejects m     O(1)
```

Total: $O(n^2)$

More Explicit Algorithm - Preferred

```
def stableMatching( men, women, men_pref_array,
                    women_pref_array):
   Initialize each person to be free (set up data structures)   O(n)
   Create inverse array for women's preferences                 O(n^2)
   while (some man is free and hasn't proposed to every woman)  O(n^2)
      Choose such a man m                                       O(1)
      w = 1st woman on m's list to whom m has not yet proposed  O(1)
      if (w is free)                                            O(1)
         assign m and w to be engaged                           O(1)
      else if (w prefers m to her fiancé m')                    O(1)  Using inverse array
         assign m and w to be engaged and m' to be free         O(1)
      else
         w rejects m                                            O(1)
   return engagements
```

Total: $O(n^2)$

A SURVEY OF COMMON RUNNING TIMES

Linear Time: O(n)

- Running time is at most a ***constant*** factor times the size of the input
- Example. Computing the maximum: Compute maximum of n numbers $a_1, \ldots, a_n$

```
max = a₁
for i = 2 to n
   if (aᵢ > max)
      max = aᵢ
```

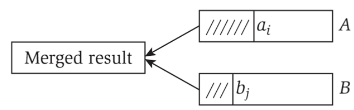
Constant work for each input (does not depend on n)

Example Linear Time: O(n)

- Merge: Combine two sorted lists $A = a_1, a_2, \ldots, a_n$ with $B = b_1, b_2, \ldots, b_n$ into sorted whole

Example Linear Time: O(n)

- Merge: Combine two sorted lists $A = a_1, a_2, \ldots, a_n$ with $B = b_1, b_2, \ldots, b_n$ into sorted whole
- Claim. Merging two lists of size *n* takes O(n) time

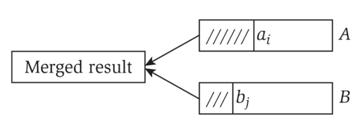



```
i = 1, j = 1
while (both lists are nonempty)
   if (a_i ≤ b_j)
      append a_i to output list and increment i
   else
      append b_j to output list and increment j

append remainder of nonempty list to output list
```


Example Linear Time: O(n)

- Merge: Combine two sorted lists $A = a_1, a_2, \ldots, a_n$ with $B = b_1, b_2, \ldots, b_n$ into sorted whole
- Claim. Merging two lists of size n takes O(n) time
- Proof. After each comparison, the length of output list increases by 1

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O(n log n) Time

- Also referred to as ***linearithmic*** time
- Arises in divide-and-conquer algorithms
  - Splitting input into equal pieces, solve recursively, combine solutions in linear time

What well-known set of algorithms has an O(n logn) running time?

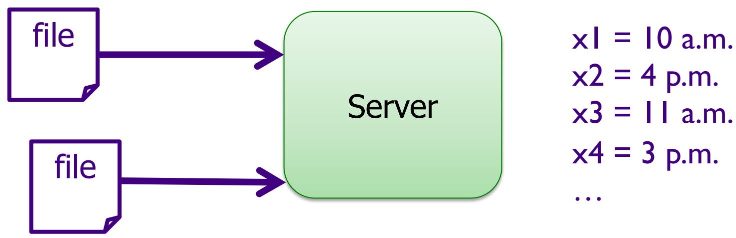
O(n log n) Time Example

- Sorting: Mergesort and heapsort are sorting algorithms that perform O(n log n) comparisons
- Mergesort
  1. Break input into equal-sized pieces
  2. Sorts each half recursively
  3. Merges sorted halves into a sorted list

Talk about the bound on running time later…

O(n log n) Time Example

- Largest empty interval. Given *n* (not necessarily ordered) time-stamps $x_1, \ldots, x_n$ at which copies of a file arrive at a server, what is largest interval of time when no copies of the file arrive?



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O(n log n) Time Example

- Largest empty interval. Given *n* (not necessarily ordered) time-stamps $x_1, \ldots, x_n$ at which copies of a file arrive at a server, what is largest interval of time when no copies of the file arrive?
- O(n log n) solution
  1. Sort time-stamps
  2. Scan sorted list in order, identifying the maximum gap between successive time-stamps

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Quadratic Time: $O(n^2)$

- Examples?

Quadratic Time: $O(n^2)$

- Examples:
  - Enumerate all pairs of elements
  - Sometimes involves nested loops (n iterations)
  - Insertion sort

Quadratic Time: $O(n^2)$

- Closest pair of points. Given a list of n points in the plane $(x_1, y_1), \ldots, (x_n, y_n)$, find the pair that is closest
- $O(n^2)$ solution. Try all pairs of points

```
min = (x1 - x2)^2 + (y1 - y2)^2
for i = 1 to n
   for j = i+1 to n
      d = (xi - xj)^2 + (yi - yj)^2
      if (d < min)
         min = d
```

don't need to take square roots

$\Omega(n^2)$ seems inevitable, but Chapter 5 has an $O(n \log n)$ solution

Polynomial Time: $O(n^k)$ Time

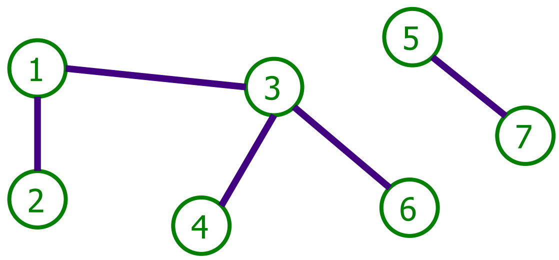
- To get all pairs, the algorithm is $O(n^2)$
- To get all triplets, the algorithm is $O(n^3)$

What is an example of an $O(n^k)$ algorithm?

All subsets of size *k*

Polynomial Time: $O(n^k)$ Time

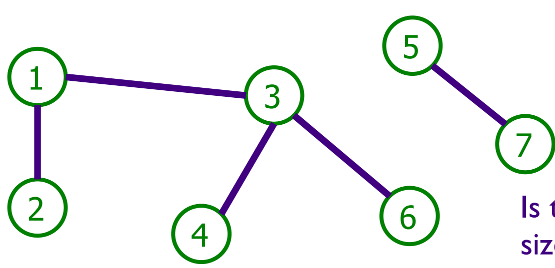
- Independent set of size *k*. Given a graph, are there *k* nodes such that no two are joined by an edge?
  - *k* is a constant



Is there an independent set of size 2? 3? 4? 5?

Polynomial Time: $O(n^k)$ Time

- Independent set of size *k*. Given a graph, are there *k* nodes such that no two are joined by an edge?
  - *k* is a constant



Is there an independent set of size 2? Yes (2-3; 1-5; 6-7; …)
3? (5-6-7; 2-3-5; …)
4? (2-4-6-7; 1-4-6-7; …)
But not 5

Polynomial Time: $O(n^k)$ Time

- Independent set of size *k*. Given a graph, are there *k* nodes such that no two are joined by an edge?
  - *k* is a constant

```
foreach subset S of k nodes
   check whether S in an independent set
   if (S is an independent set)
      report S is an independent set
```

- $O(n^k)$ solution
  1. Enumerate all subsets of k nodes

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!} = \frac{n\,(n-1)\,(n-2)\ldots(n-k+1)}{k\,(k-1)\,(k-2)\ldots(2)\,(1)} \leq \frac{n^k}{k!}$$

  1. Check whether S is an independent set = $O(k^2)$.

$O(k^2\, n^k / k!) = O(n^k)$

poly-time for k=17 but not practical

Exponential Time

- Independent set. Given a graph, what is the *maximum size* of an independent set?
- $O(n^2\, 2^n)$ solution. Enumerate all subsets

```
S* = ϕ
foreach subset S of nodes
   check whether S in an independent set
   if (S is largest independent set seen so far)
      S* = S
```

O(log n) Time

- ***Sublinear*** time
- Know any algorithms that take O(log n) time?

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O(log n) Time

- Example: Binary search
- Often requires some pre-processing or data structure that allows cheaper "querying" than *n* time

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Summary of Running Times

Running Time	Example
O(log n)	Generally dividing problem in half on each iteration
O(n)	Operate on each input value
O(n log n)	Divide and conquer
$O(n^2)$	Operate on each pair of inputs
O(n!)	Operate on each permutation of inputs

MORE COMPLEX DATA STRUCTURES

Improving Running Times

After overcoming higher-level obstacles, lower-level **implementation details** can **improve runtime**.

PRIORITY QUEUES

Priority Queues

- Elements have a ***priority*** or *key*
- Each time select an element from the priority queue, want the one with *highest* priority
- More formally…
  - Maintains a set of elements *S*
    - Each element $v \in S$ has a $\text{key}(v)$ for its priority
      - Smaller keys represent higher priorities
  - Application Programming Interface
    - Add, delete elements
    - Select element with smallest key

Key	2	4	5	6	9	20
Value	3542	5143	8712	1264	9123	5954

Key ← Priority
Value ← Process id

(Not implementation, just how to envision)

Motivating Example: Scheduling Processes

Key	2	4	5	6	9	20
Value	3542	5143	8712	1264	9123	5954

Key ← Priority
Value ← Process id

- Each process has a priority or urgency
- Processes do not arrive in priority order
- **Goal**: run process with highest priority

Using a Priority Queue (PQ)

- PQ API:
  - Add an element with a given key (i.e., priority)
  - Delete an element with a given priority
  - Select element with smallest key/highest priority
  - Get the number of elements in PQ

Given a list of numbers, how could you use a PQ to sort that list of numbers?

Priority Queues for Sorting

1. Add elements into PQ with the number's value as its priority
2. Then extract the smallest number *until* done
   - Come out in sorted order

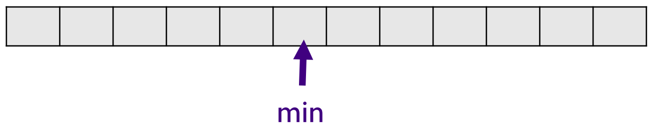
Sorting *n* numbers takes O(n logn) time

What is the goal running time for our PQ's operations? **O(logn)**

Already know our "loops" will be O(n)

Implementing a Priority Queue

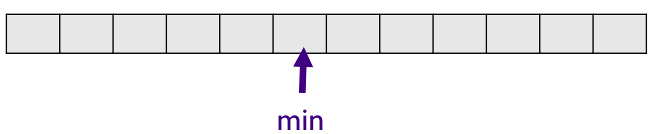
- Consider an *unordered* list, where there is a pointer to minimum



- How difficult (i.e., expensive) is
  - Adding new elements?
  - Extraction?

Implementing a Priority Queue

- Consider an *unordered* list, where there is a pointer to minimum



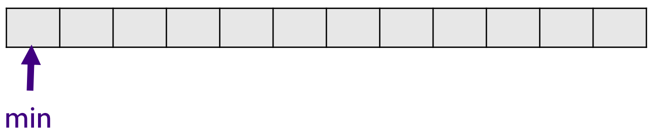
- How difficult (i.e., expensive) is
  - Adding new elements? *easy (O(1))*
  - Extraction? *difficult*
    - Need to find "new" minimum: O(n)

What is the running time for sorting using the PQ in this case?

$O(n^2)$

Implementing a Priority Queue

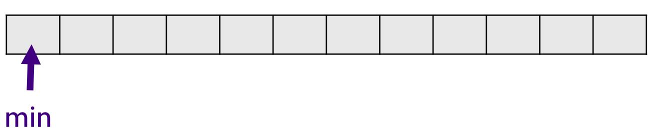
- Consider a *sorted* list where min is at the beginning



- Should you use an array or linked list?
- How difficult is
  - Adding new elements?
  - Extraction?

Implementing a Priority Queue

- Consider a sorted list where min is at the beginning



- Should you use an array or linked list?
- How difficult is
  - Adding new elements? *difficult (insertion) – O(n)*
  - Extraction? *Easy – O(1)*

What is the running time for sorting using the PQ in this case?

$O(n^2)$

Looking Ahead

- Wiki due Monday
  - 2.3, 2.4
- Problem Set 2 due Friday

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