Objectives

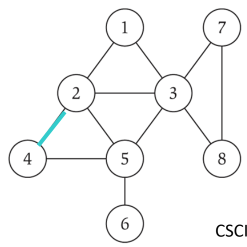
- Graphs
- Graph Connectivity, Traversal
- BFS & DFS Implementations, Analysis

Review

- What is a heap?
  - When is it useful?
- What is a graph?
  - What are two ways to implement a graph?
  - What are their space costs?
  - What are the operations that can be performed on them?
  - What is the [time] cost of those operations?

Review: Graph Representation: Adjacency Matrix

- n × n matrix with $A_{uv} = 1$ if (u, v) is an edge
  - Two representations of each edge (symmetric matrix)
  - Space: $\Theta(n^2)$
  - Checking if (u, v) is an edge: $\Theta(1)$ time
  - Identifying all edges: $\Theta(n^2)$ time

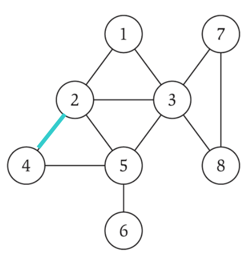


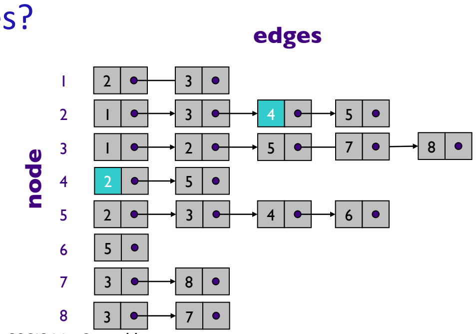
	1	2	3	4	5	6	7	8
1	0	1	1	0	0	0	0	0
2	1	0	1	1	1	0	0	0
3	1	1	0	0	1	0	1	1
4	0	1	0	0	1	0	0	0
5	0	1	1	1	0	1	0	0
6	0	0	0	0	1	0	0	0
7	0	0	1	0	0	0	0	1
8	0	0	1	0	0	0	1	0

Graph Representation: Adjacency List

- Node indexed array of lists
  - Two representations of each edge
  - Space?
  - Checking if (u, v) is an edge?
  - Identifying all edges?

What are the extremes?

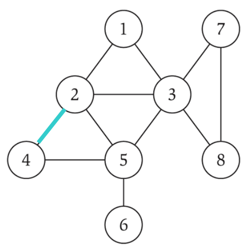


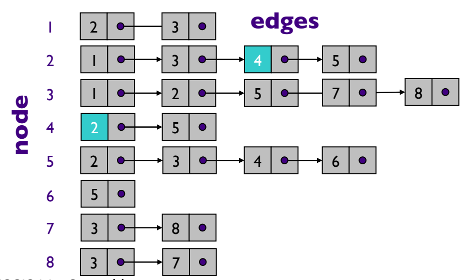


Graph Representation: Adjacency List

- Node indexed array of lists
  - Two representations of each edge
  - Space = $2m + n = O(m + n)$
  - Checking if (u, v) is an edge takes $O(\deg(u))$ time
  - Identifying all edges takes $\Theta(m + n)$ time

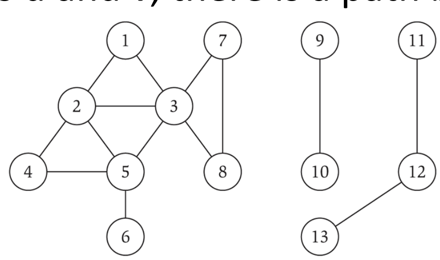
degree = number of neighbors of u





Paths and Connectivity

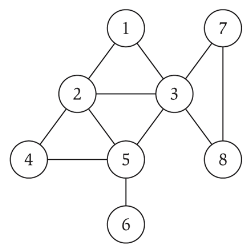
- Def. A ***path*** in an undirected graph G = (V, E) is a sequence P of nodes $v_1, v_2, \ldots, v_{k-1}, v_k$
  - Each consecutive pair v_i, v_{i+1} is joined by an edge in E
- Def. A path is ***simple*** if all nodes are *distinct*
- Def. An undirected graph is ***connected*** if ∀ pair of nodes u and v, there is a path between u and v



- *Short path*
- *Distance*

Cycles

- Def. A ***cycle*** is a path v_1, v_2, …, v_{k-1}, v_k in which $v_1 = v_k$, $k > 3$, and the first k-1 nodes are all distinct

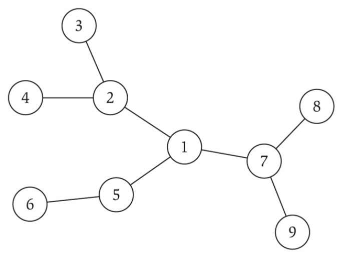


cycle C = 1-2-4-5-3-1

TREES

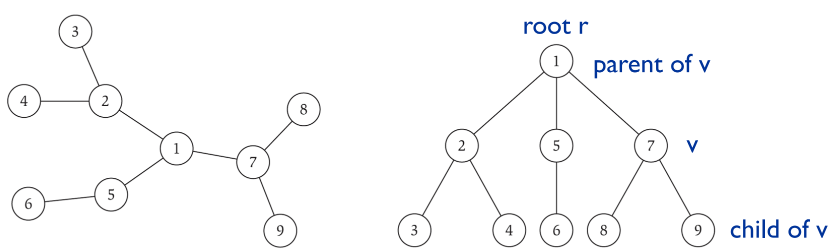
Trees

- Def. An undirected graph is a ***tree*** if it is connected and does not contain a cycle
- Simplest connected graph
  - Deleting any edge from a tree will disconnect it



Rooted Trees

- Given a tree T, choose a root node *r* and orient each edge away from *r*
- Models hierarchical structure

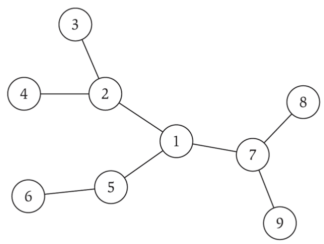


a tree the same tree, rooted at 1

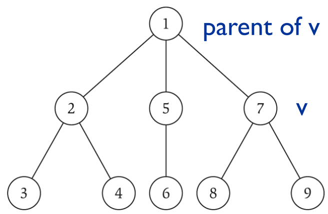
Why *n-1* edges?

Rooted Trees

- Why *n-1* edges?
  - Each non-root node has an edge to its parent



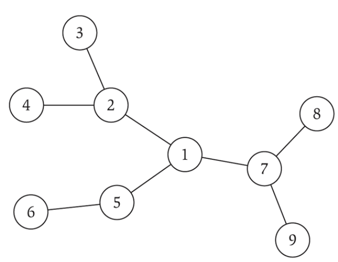
a tree



the same tree, rooted at 1

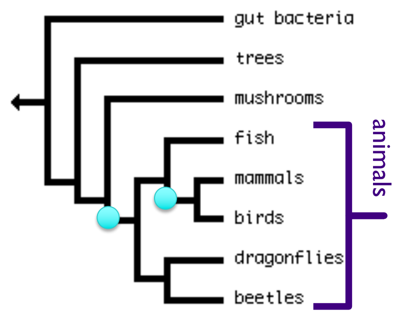
Trees

- Theorem. Let G be an undirected graph on *n* nodes. Any two of the following statements imply the third:
  - G is connected
  - G does not contain a cycle
  - G has *n*-1 edges



Phylogeny Trees

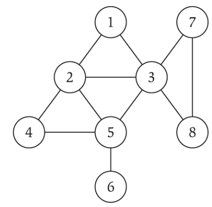
- Describe evolutionary history of species
  - mammals and birds share a common ancestor that they do not share with other species
  - all animals are descended from an ancestor not shared with mushrooms, trees, and bacteria



GRAPH CONNECTIVITY & TRAVERSAL

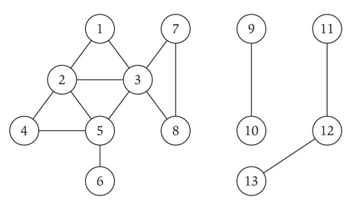
Connectivity

- **s-t connectivity problem.** Given nodes *s* and *t*, is there a path between *s* and *t*?
- **s-t shortest path problem.** Given nodes *s* and *t*, what is the length of the shortest path between *s* and *t*?
- Applications
  - Facebook
  - Maze traversal
  - Kevin Bacon number
  - Spidering the web
  - Fewest number of hops in a communication network



Application: Connected Component

- Find all nodes ***reachable*** from *s*

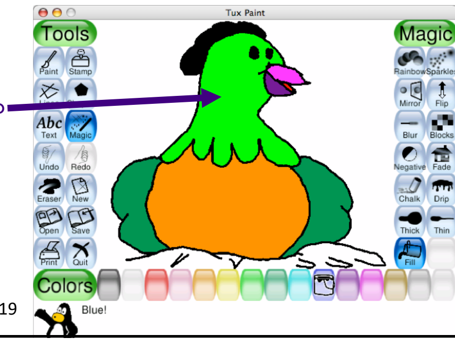


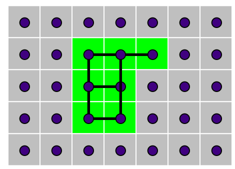
- Connected component containing node 1 is { 1, 2, 3, 4, 5, 6, 7, 8 }

Application: Flood Fill

- Given lime green pixel in an image, change color of entire blob of neighboring lime pixels to blue
  - Node: pixel
  - Edge: two neighboring lime pixels
  - Blob: connected component of lime pixels

recolor lime green blob to blue





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Application: Flood Fill

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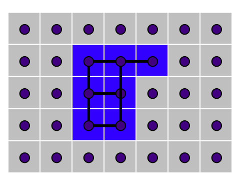

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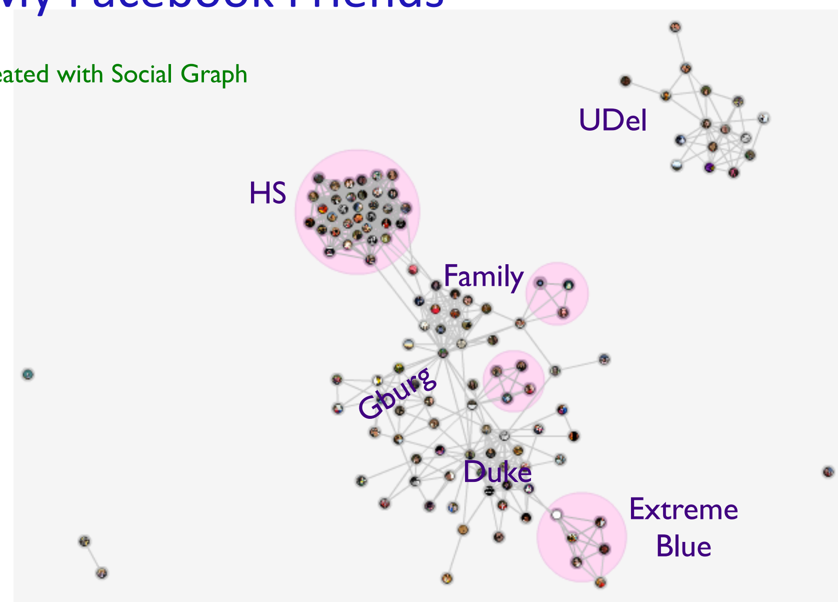
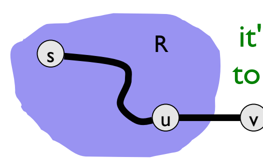
My Facebook Friends

Created with Social Graph



A General Algorithm

```
R will consist of nodes to which s has a path
R = {s}
while there is an edge (u,v) where u∈R and v∉R
      add v to R
```



it's safe to add v

- *R* will be the **connected component** containing s
- Algorithm is underspecified

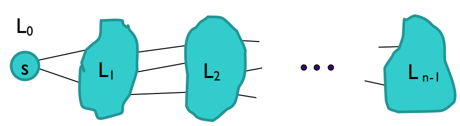
In what order should we consider the edges?

Possible Orders

- Breadth-first
- Depth-first

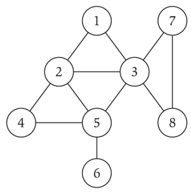
Breadth-First Search

- **Intuition**. Explore outward from *s* in all possible directions (edges), adding nodes one "layer" at a time
- **Algorithm**



  - L_0 = { s }
  - L_1 = all neighbors of L_0
  - L_2 = all nodes that have an edge to a node in L_1 and do not belong to L_0 or L_1
  - L_{i+1} = all nodes that have an edge to a node in L_i and do not belong to an earlier layer

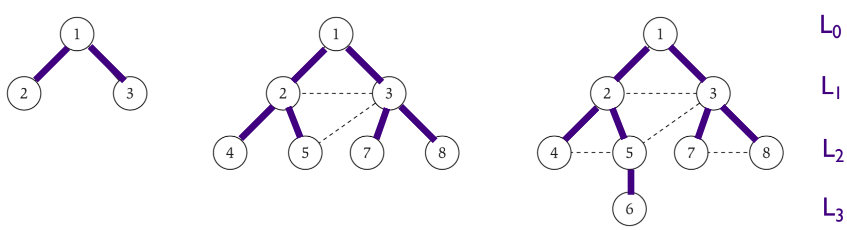
Run BFS on This Graph



s = 1

Example of Breadth-First Search

s = 1

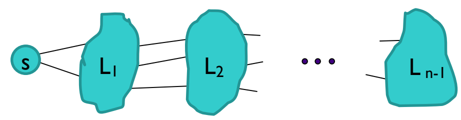


Creates a tree

-- is a node in the graph that is not in the tree

Breadth-First Search

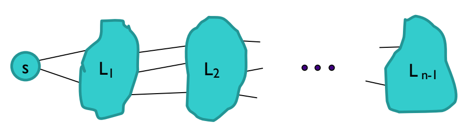
- **Theorem.**
  For each i, L_i consists of all nodes at distance exactly i from s.
  There is a path from s to t iff t appears in some layer.



- What does this theorem mean?
- Can we determine the distance between s and t?

Breadth-First Search

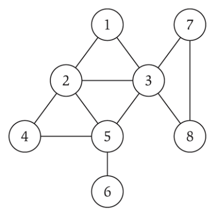
- **Theorem.** For each i, L_i consists of all nodes at distance exactly i from s. There is a path from s to t iff t appears in some layer.
  - Length of shortest path to t from s, is the i from L_i
  - All nodes ***reachable*** from s are in L_1, L_2, …, L_{n-1}



Breadth-First Search

- **Property**. Let T be a BFS tree of G = (V, E), and let (x, y) be an edge of G.
  Then the level of x and y *differ* by *at most* 1.

G:

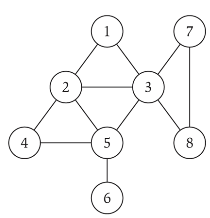


If x is in L_i,
then y must be in ???

Breadth-First Search

- **Property**. Let T be a BFS tree of G = (V, E), and let (x, y) be an edge of G. Then the level of x and y *differ* by *at most* 1.

G:



If x is in L_i,
then y must be in

- **L_{i-1}**: y was reached before x
- **L_i**: a common parent of x and y was reached first
- **L_{i+1}**: y will be added in the next layer

Connected Component: BFS

- Find all nodes ***reachable*** from *s*

In general….

```
R will consist of nodes to which s has a path
R = {s}
while there is an edge (u,v) where u∈R and v∉R
      add v to R
```

In what order does BFS consider edges?

Connected Component: BFS vs DFS

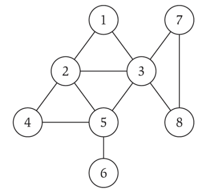
- Find all nodes ***reachable*** from *s*

In general….

```
R will consist of nodes to which s has a path
R = {s}
while there is an edge (u,v) where u∈R and v∉R
      add v to R
```

- Theorem. Upon termination, R is the connected component containing s
  - BFS = explore in order of distance from s
  - DFS = explore until hit "deadend"

Depth-First Search

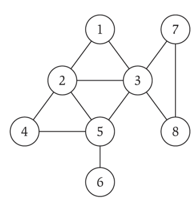


- Need to keep track of where you've been
- When reach a "dead-end" (already explored all neighbors), backtrack to node with unexplored neighbor
- **Algorithm**:

```
DFS(u):
   Mark u as "Explored" and add u to R
   For each edge (u, v) incident to u
      If v is not marked "Explored" then
         DFS(v)
```


Depth-First Search

- How does DFS work on this graph?
  - Starting from node 1



Looking Ahead

- Monday, 11:59 p.m.: journal - Chapter 2.5, 3.1
- Friday: Problem Set 3 due