Objectives

- Dynamic Programming
  - Knapsack
  - Sequence Alignment

Review

- What is the segmented least squares problem?
  - What was our solution?
- What is the knapsack problem?
  - What was our solution (so far)?

Segmented Least Squares: Algorithm Analysis

How do we find the *solution*?

```
INPUT: n, p1,…,pn , c

Segmented-Least-Squares()
   M[0] = 0
   e[0][0] = 0
   for j = 1 to n
      for i = 1 to j
         e[i][j] = least square error for the
                   segment pi,…, pj

   for j = 1 to n
      M[j] = min 1 ≤ i ≤ j (e[i][j] + c + M[i-1])

   return M[n]
```

can be improved to $O(n^2)$ by pre-computing various statistics

$O(n^3)$

$O(n^2)$

- Bottleneck: computing e(i, j) for $O(n^2)$ pairs, O(n) per pair using previous formula

Post-Processing: Finding the Solution

```
FindSegments(j):
   if j = 0:
      output nothing
   else:
      Find an i that minimizes ei,j + c + M[i-1]
      Output the segment {pi, …, pj}
      FindSegments(i-1)
```

Cost? $O(n^2)$

Call as: `FindSegments(n)`

Knapsack Problem

- Given *n* objects and a "knapsack"
- Item *i* weighs $w_i > 0$ kilograms and has value $v_i > 0$
  - Example: jobs require w_i time
- Knapsack has capacity of *W* kilograms
  - Example: *W* is time interval that resource is available

Goal: fill knapsack so as to maximize total value

W = 11

Item	Value	Weight
1	1	1
2	6	2
3	18	5
4	22	6
5	28	7

Dynamic Programming: Adding a New Variable

- Def. OPT(i, **w**) = max profit subset of items 1, …, i with weight limit **w**
  - Case 1: OPT does not select item *i*
    - OPT selects best of { 1, 2, …, i-1 } using weight limit ***w***
  - Case 2: OPT selects item *i*
    - new weight limit = $w - w_i$
    - OPT selects best of { 1, 2, …, i–1 } using new weight limit, $\boldsymbol{w - w_i}$

$$OPT(i,w) = \begin{cases} 0 & \text{if } i = 0 \\ OPT(i-1,w) & \text{if } w_i > w \\ \max\{OPT(i-1,w),\ v_i + OPT(i-1,w-w_i)\} & \text{otherwise} \end{cases}$$

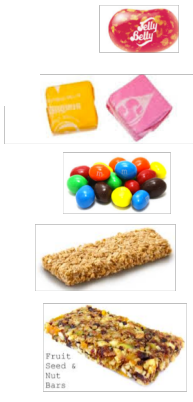
Knapsack Problem: Bottom-Up

- Fill up an n-by-W array

```
Input: W, N, w1,…,wN, v1,…,vN

for w = 0 to W    # base case: no items, so value is 0
   M[0, w] = 0

for i = 1 to N     # for all items
   for w = 0 to W  # for all possible weights
      if wi > w :  # item's weight is more than available
         M[i, w] = M[i-1, w]
      else
         M[i, w] = max{ M[i-1, w], vi + M[i-1, w-wi] }

return M[n, W]
```


Knapsack Input

W = 11



Item	Value	Weight
1	1	1
2	6	2
3	18	5
4	22	6
5	28	7

Knapsack Algorithm

Represents weight in knapsack

W → # of entries: W + 1

i → # of entries: n + 1

	0	1	2	3	4	5	6	7	8	9	10	11
{ }												
{ 1 }												
{ 1, 2 }												
{ 1, 2, 3 }												
{ 1, 2, 3, 4 }												
{ 1, 2, 3, 4, 5 }												

Represents item id

What happens when we consider the most recent item?

OPT:

Solution =

W = 11

Item	Value	Weight
1	1	1
2	6	2
3	18	5
4	22	6
5	28	7

Knapsack Algorithm

W → # of entries: W + 1

i → # of entries: n + 1

	0	1	2	3	4	5	6	7	8	9	10	11
{ }	0	0	0	0	0	0	0	0	0	0	0	0
{ 1 }												
{ 1, 2 }												
{ 1, 2, 3 }												
{ 1, 2, 3, 4 }												
{ 1, 2, 3, 4, 5 }												

OPT:

Solution =

W = 11

Item	Value	Weight
1	1	1
2	6	2
3	18	5
4	22	6
5	28	7

Knapsack Algorithm

i = 1

W + 1

n + 1

	0	1	2	3	4	5	6	7	8	9	10	11
ϕ	0	0	0	0	0	0	0	0	0	0	0	0
{ 1 }	0	1	1	1	1	1	1	1	1	1	1	1
{ 1, 2 }	0											
{ 1, 2, 3 }	0											
{ 1, 2, 3, 4 }	0											
{ 1, 2, 3, 4, 5 }	0											

OPT:
Solution =

W = 11

Item	Value	Weight
1	1	1
2	6	2
3	18	5
4	22	6
5	28	7

Knapsack Algorithm

i = 2

W + 1

n + 1

	0	1	2	3	4	5	6	7	8	9	10	11
ϕ	0	0	0	0	0	0	0	0	0	0	0	0
{ 1 }	0	1	1	1	1	1	1	1	1	1	1	1
{ 1, 2 }	0	1	6	7	7	7	7	7	7	7	7	7
{ 1, 2, 3 }	0											
{ 1, 2, 3, 4 }	0											
{ 1, 2, 3, 4, 5 }	0											

OPT:
Solution =

W = 11

Item	Value	Weight
1	1	1
2	6	2
3	18	5
4	22	6
5	28	7

Knapsack Algorithm

i = 3

W + 1 →

n + 1 ↓

	0	1	2	3	4	5	6	7	8	9	10	11
ϕ	0	0	0	0	0	0	0	0	0	0	0	0
{ 1 }	0	1	1	1	1	1	1	1	1	1	1	1
{ 1, 2 }	0	1	6	7	7	7	7	7	7	7	7	7
{ 1, 2, 3 }	0	1	6	7	7	18	19	24	25	25	25	25
{ 1, 2, 3, 4 }	0											
{ 1, 2, 3, 4, 5 }	0											

OPT:
Solution =

W = 11

Item	Value	Weight
1	1	1
2	6	2
3	18	5
4	22	6
5	28	7

Knapsack Algorithm

i = 4

W + 1 →

n + 1 ↓

	0	1	2	3	4	5	6	7	8	9	10	11
ϕ	0	0	0	0	0	0	0	0	0	0	0	0
{ 1 }	0	1	1	1	1	1	1	1	1	1	1	1
{ 1, 2 }	0	1	6	7	7	7	7	7	7	7	7	7
{ 1, 2, 3 }	0	1	6	7	7	18	19	24	25	25	25	25
{ 1, 2, 3, 4 }	0	1	6	7	7	18	22	24	28	29	29	40
{ 1, 2, 3, 4, 5 }	0											

OPT:
Solution =

W = 11

Item	Value	Weight
1	1	1
2	6	2
3	18	5
4	22	6
5	28	7

Knapsack Algorithm

W + 1

i = 5

n + 1

	0	1	2	3	4	5	6	7	8	9	10	11
ϕ	0	0	0	0	0	0	0	0	0	0	0	0
{ 1 }	0	1	1	1	1	1	1	1	1	1	1	1
{ 1, 2 }	0	1	6	7	7	7	7	7	7	7	7	7
{ 1, 2, 3 }	0	1	6	7	7	18	19	24	25	25	25	25
{ 1, 2, 3, 4 }	0	1	6	7	7	18	22	24	28	29	29	40
{ 1, 2, 3, 4, 5 }	0	1	6	7	7	18	22	28	29	34	35	40

Item	Value	Weight
1	1	1
2	6	2
3	18	5
4	22	6
5	28	7

W = 11

OPT:
Solution =

What is the optimal solution?

Knapsack Algorithm

W + 1

n + 1

	0	1	2	3	4	5	6	7	8	9	10	11
ϕ	0	0	0	0	0	0	0	0	0	0	0	0
{ 1 }	0	1	1	1	1	1	1	1	1	1	1	1
{ 1, 2 }	0	1	6	7	7	7	7	7	7	7	7	7
{ 1, 2, 3 }	0	1	6	7	7	18	19	24	25	25	25	25
{ 1, 2, 3, 4 }	0	1	6	7	7	18	22	24	28	29	29	40
{ 1, 2, 3, 4, 5 }	0	1	6	7	7	18	22	28	29	34	35	40

Item	Value	Weight
1	1	1
2	6	2
3	18	5
4	22	6
5	28	7

W = 11

OPT: 40 = 22 + 18
Solution={4, 3}

Why was it helpful to order the items by their weights?

Analyzing Solution

How do we figure out the optimal *solution*?

```
Input: W, n, w1,…,wn, v1,…,vn

for w = 0 to W
   M[0, w] = 0

for k = 1 to n
   M[k, 0] = 0

for i = 1 to n      # for all items
   for w = 1 to W   # for all possible weights
      if wi > w :   # item's weight is more than available
         M[i, w] = M[i-1, w]
      else
         M[i, w] = max{ M[i-1, w], vi + M[i-1, w-wi] }

return M[n, W]
```

Costs?

Analyzing Solution

```
Input: W, n, w1,…,wn, v1,…,vn

for w = 0 to W      O(W)
   M[0, w] = 0

for k = 1 to n      O(n)
   M[k, 0] = 0                                   O(n W)

for i = 1 to n      # for all items
   for w = 1 to W   # for all possible weights
      if wi > w :   # item's weight is more than available
         M[i, w] = M[i-1, w]
      else
         M[i, w] = max{ M[i-1, w], vi + M[i-1, w-wi] }

return M[n, W]
```

Knapsack Problem: Running Time

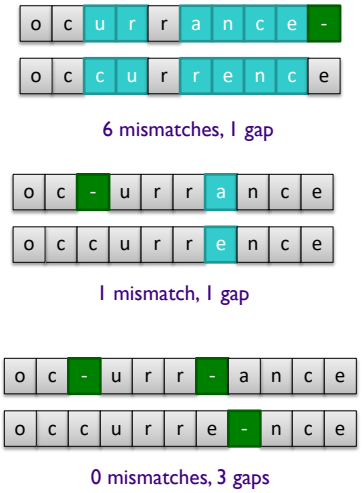
- Running time. $\Theta(n\ W)$
  - **Not** polynomial in input size!
  - "Pseudo-polynomial"
    - Reasonably efficient when W is reasonably small
  - Decision version of Knapsack is NP-complete [Chapter 8]
- Knapsack approximation algorithm. There exists a polynomial algorithm that produces a feasible solution that has value within 0.01% of optimum. [Section 11.8]

STRING SIMILARITY

String Similarity

- How similar are two strings?
  - ocurrance
  - occurrence
- Measurements
  - Gap (-): add a letter
  - Mismatch

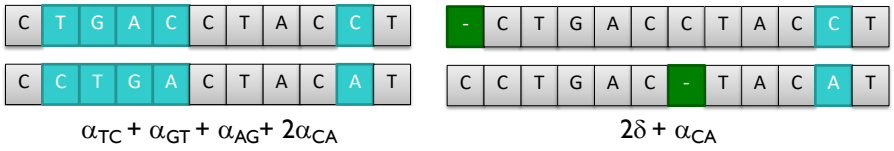
- Which is the best alignment?
- When would it be helpful to know how similar two strings are?



Edit Distance

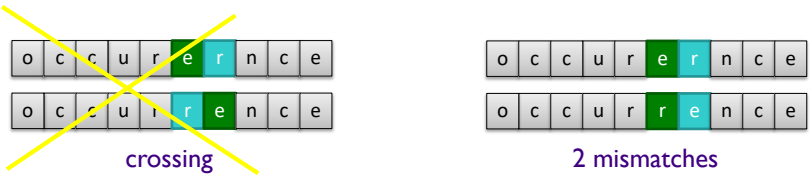
- [Levenshtein 1966, Needleman-Wunsch 1970]
  - Gap penalty: δ
  - Mismatch penalty: α_{pq}
    - If p and q are the same, then mismatch penalty is 0
  - **Cost** = sum of gap and mismatch penalties

Parameters allow us to tweak cost



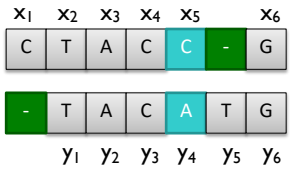
Sequence Alignment

- Goal: Given two strings $X = x_1 x_2 \ldots x_m$ and $Y = y_1 y_2 \ldots y_n$ find alignment of minimum cost
- An *alignment* M is a set of ordered pairs x_i-y_j such that each item occurs in at most one pair and **no** crossings
- The pair x_i-y_j and $x_{i'}$-$y_{j'}$ *cross* if i < i', but j > j'.



Sequence Alignment Example

- X = CTACCG
- Y = TACATG
- Solution: M = x_2-y_1 , x_3-y_2, x_4-y_3, x_5-y_4 , x_6-y_6



$$\text{cost}(M) = \underbrace{\sum_{(x_i, y_j) \in M} \alpha_{x_i y_j}}_{\text{mismatch}} + \underbrace{\sum_{i : x_i \text{ unmatched}} \delta + \sum_{j : y_j \text{ unmatched}} \delta}_{\text{gap}}$$

Recall: mismatch penalty is 0 if x_i and y_j are the same

Looking Ahead

- Wiki: 6 – 6.4
- PS8 - Friday