Objectives

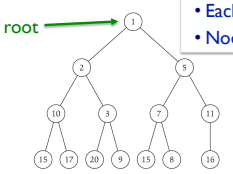
- Wrap up: Implementing a PQ
- Data structure: Graphs

Review

- What is a priority queue?
- What is a heap?
  - Properties
  - Implementation
- What is the process for finding the smallest element in a heap?
- What is the process for adding to a heap?

Review: Heap Defined

- Combines benefits of sorted array and list
- Balanced binary tree



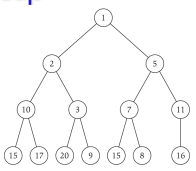
- Each node has *at most* 2 children
- Node value is its *key*

Heap order: each node's key is at least as large as its parent's

Note: **not** a binary search tree

Review: Implementing a Heap

- Option 1: Use pointers
  - Each node keeps
    - Element it stores (key)
    - 3 pointers: 2 children, parent
- Option 2: No pointers
  - Requires knowing upper bound on *n*
  - For node at position *i*
    - left child is at *2i*
    - right child is at *2i+1*

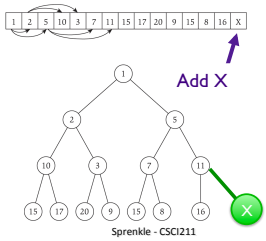


Review: Implementing a Heap

- Finding the minimal element
  - First element
  - O(1)

Implementing a Heap: Operations

- Adding an element?
  - Could add element to last position
    - What are possible scenarios?



Implementing a Heap: Operations

- Adding an element?
  - Could add element to last position
    - What are possible scenarios?
      - Heap is no longer balanced
      - Something that is almost a heap but a little off
      - Need `Heapify-up` procedure to fix our heap

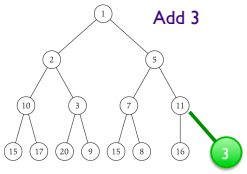
Heapify-Up

Heap — Position where node added

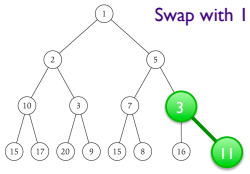
```
Heapify-up(H, i):
  if i > 1 then
      j=parent(i)=floor(i/2)
      if key[H[i]] < key[H[j]] then
        swap array entries H[i] and H[j]
        Heapify-up(H, j)
```

- Why does this algorithm work?
- What is the intuition?

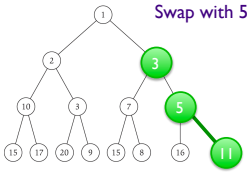
Practice: Heapify-Up



Practice: Heapify-Up



Practice: Heapify-Up



Heapify-Up

- Claim. Assuming array H is almost a heap with key of `H[i]` too small, `Heapify-Up` fixes the heap property in O(log i) time
  - Can insert a new element in a heap of *n* elements in O(log n) time

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- Proof. By induction
  - If i=1 …

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  - If i=1, is already a heap → O(1)
  - If i>1, …

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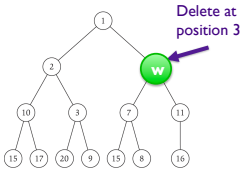
  - Can insert a new element in a heap of *n* elements in O(log n) time
- Proof. By induction
  - If i=1, is already a heap → O(1)
  - If i>1,
    - Swaps are O(1)
    - Swaps continue up to root (max) → log i

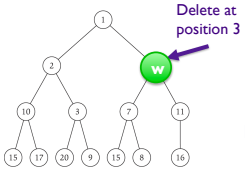
Deleting an Element



Deleting an Element

- Delete at position *i*
- Removing an element:
  - Messes up heap order
  - Leaves a "hole" in the heap
- Not as straightforward as `Heapify-Up`
- Algorithm
  1. Fill in element where hole was
     - Patch hole: move n^{th} element into i^{th} spot
  2. Adjust heap to be in order
     - At position *i* because moved n^{th} item up to *i*

Deleting an Element

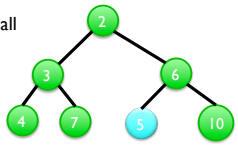


Example of OK:
11 deleted, replaced by 16

- Two "bad" possibilities: element *w* is
  - Too small: violation is between it and parent → `Heapify-Up`
  - Too big: with one or both children → `Heapify-Down` (example: w becomes 12)

Deleting an Element

Example where new key is too small



- Delete 9
- Replace with 5
- But 5 < 6, so need to `Heapify-Up`

Heapify-Down

```
Heapify-down(H, i):
   n = length(H)
   if 2i > n then
      Terminate with H unchanged
   else if 2i < n then
      left=2i and right=2i+1
      j be index that minimizes
            key[H[left]] and key[[H[right]]
   else if 2i = n then
      j=2i

   if key[H[j]] < key[H[i]] then
      swap array entries H[i] and H[j]
      Heapify-down(H, j)
```

Why can we stop?

Heapify-Down

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Heapify-down(H, i):
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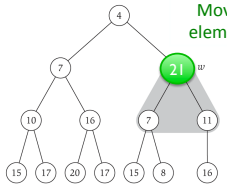
```

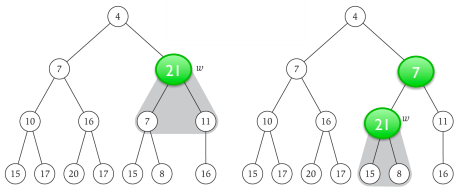
i is a leaf – nowhere to go

Practice: Heapify-Down

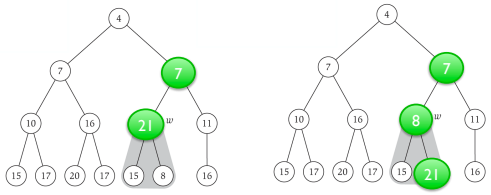


Moved 21 to where element was removed

Practice: Heapify-Down



Practice: Heapify-Down



Runtime of Heapify-Down?

```
Heapify-down(H, i):
   n = length(H)
   if 2i > n then
      Terminate with H unchanged
   else if 2i < n then
      left=2i and right=2i+1
      j be index that minimizes    O(1)
            key[H[left]] and key[[H[right]]
   else if 2i = n then
      j=2i

   if key[H[j]] < key[H[i]] then
      swap array entries H[i] and H[j] O(1)
      Heapify-down(H, j)
```

Num swaps: O(log n)

Implementing Priority Queues with Heaps

Operation	Description	Run Time
StartHeap(N)	Creates an empty heap that can hold N elements	
Insert(v)	Inserts item v into heap	
FindMin()	Identifies minimum element in heap but does not remove it	
Delete(i)	Deletes element in heap at position i	
ExtractMin()	Identifies and deletes an element with minimum key from heap	

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ExtractMin()	Identifies and deletes an element with minimum key from heap	O(log n)

Putting It All Together…

1. Add elements into PQ with the number's value as its priority
2. Then extract the smallest number until done
   - Come out in sorted order

What is the running time of sorting numbers using a PQ implemented with a **heap**?

O(n log n)

Comparing Data Structures

Operation	Heap	Unsorted List	Sorted List
Start(N)		O(1)	O(1)
Insert(v)		O(1)	O(n)
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Comparing Data Structures

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Start(N)	O(N)	O(1)	O(1)
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ExtractMin()	O(log n)	O(n)	O(1)

Not covered in class, but in book

Additional Heap Operations

- Access elements in PQ by "name"

Key	2	4	5	6	9	20
Value	3542	5143	8712	1264	9123	5954

Priority
Process id

  - Maintain additional array `Position` that stores current position of each element in heap

- Operations:
  - Delete(Position[v])
    - Does not increase overall running time
  - ChangeKey(v, α)
    - Changes key of element v to α
    - Identify position of element v in array (`Position` array)
    - Change key, heapify

Looking Ahead

- Problem Set 2 due Friday
- Office Hours: Meeting from 3-4 tomorrow
  - Still available from 2:30-3, 4-4:30 (another meeting at 4:30)
  - Email me about additional times