Objectives

- Graph Application: Bipartite Graphs
- Directed Graphs
- Topological Orderings of DAGs

Review

- What do we know about graphs?

Review

- What do we know about graphs?
  - Representation: Adjacency List, Space O(n+m)
  - Connectivity
    - BFS, DFS – O(n+m)

Set of All Connected Components

- How can we find set of all connected components of graph?

```
R* = set of connected components (a set of sets)
while there is a node that does not belong to R*
    select s not in R*
    R = {s}
    while there is an edge (u,v) where u∈R and v∉R
        add v to R
    Add R to R*
```

Imprecision in the running time of inner loop: O(m+n)

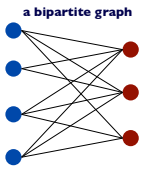
But that's m and n of the *connected* component, let's say m_i and n_i . $\sum_i O(m_i + n_i) = O(m+n)$

Where *i* is the subscript of the connected component

BIPARTITE GRAPHS

Bipartite Graphs

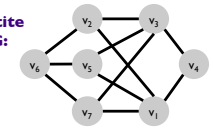
- Def. An undirected graph G = (V, E) is ***bipartite*** if the nodes can be colored red or blue such that every edge has one red and one blue end
  - Generally: vertices divided into sets X and Y
- Applications:
  - Stable marriage:
    - men = red, women = blue
  - Scheduling:
    - machines = red, jobs = blue



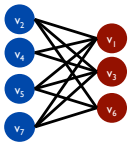
Testing Bipartiteness

- Given a graph G, is it bipartite?
- Many graph problems become:
  - Easier if underlying graph is bipartite (e.g., matching)
  - Tractable if underlying graph is bipartite (e.g., independent set)
- Before designing an algorithm, need to understand structure of bipartite graphs

a bipartite graph G:



another drawing of G:

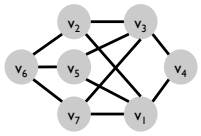


How Can We Determine if a Graph is Bipartite?

- Given a connected graph

Why connected?

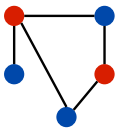
1. Color one node red
   - Doesn't matter which color (Why?)
   - What should we do next?



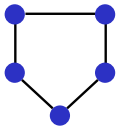
- How will we know when we're finished?
- What does this process sound like?

An Obstruction to Bipartiteness

- Lemma. If a graph G is bipartite, it cannot contain an odd-length cycle.



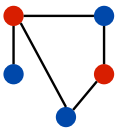
bipartite (2-colorable)



not bipartite (not 2-colorable)

An Obstruction to Bipartiteness

- Lemma. If a graph G is bipartite, it cannot contain an odd-length cycle.
- Pf. Not possible to 2-color the odd cycle, let alone G.



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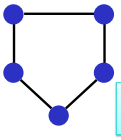


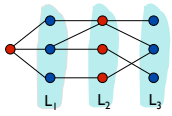
not bipartite (not 2-colorable)

If find an odd cycle, graph is NOT bipartite

How Can We Determine if a Graph is Bipartite?

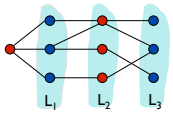
- Given a connected graph
  - Color one node red
    - Doesn't matter which color (Why?)
  - What should we do next?
- How will we know that we're finished?
- What does this process sound like?
  - BFS: alternating colors, layers

How can we implement the algorithm?



Implementing Algorithm

- Modify BFS to have a Color array
- When add v to list L[i+1]
  - Color[v] = red if i+1 is even
  - Color[v] = blue if i+1 is odd



What is the running time of this algorithm?

Marks a change in how we think about algorithms
Starting to apply known algorithms to solve new problems

Original BFS Implementation

```
BFS(s, G):
    Discovered[v] = false, for all v                    n
    Discovered[s] = true
    L[0] = {s}
    layer counter i = 0
    BFS tree T = {}
    while L[i] != {}                                    At most n
        L[i+1] = {}
        For each node u ∈ L[i]
            Consider each edge (u,v) incident to u      Σ_{u∈V} deg(u) = 2m
            if Discovered[v] == false then              O(deg(u))
                Discovered[v] = true
                Add edge (u, v) to tree T
                Add v to the list L[i + 1]
        i+=1
```

→ O(n+m)

Implementing Algorithm

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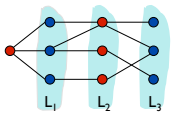


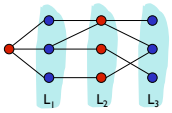
What is the running time of this algorithm? **O(n+m)**

Marks a change in how we think about algorithms
Starting to apply known algorithms to solve new problems

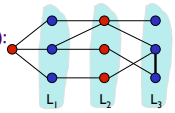
Analyzing Algorithm's Correctness

- Lemma. Let G be a connected graph, and let L_0, …, L_k be the layers produced by BFS starting at node *s*. Exactly one of the following holds:
  - (i) No edge of G joins two nodes of the same layer
    - ⇨ G is bipartite
  - (ii) An edge of G joins two nodes of the same layer
    - ⇨ G contains an odd-length cycle and hence is not bipartite

Case (i):



Case (ii):



Analyzing Algorithm's Correctness

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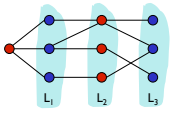
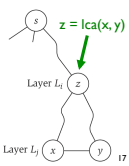
  - (i) No edge of G joins two nodes of the same layer
    - ⇨ G is bipartite
- Pf. (i)
  - Suppose no edge joins two nodes in the same layer
  - Implies all edges join nodes on adjacent level
  - Bipartition
    - red = nodes on odd levels
    - blue = nodes on even levels



Analyzing Algorithm's Correctness

- Lemma. Let G be a connected graph, and let L_0, …, L_k be the layers produced by BFS starting at node *s*. Exactly one of the following holds:
  - (ii) An edge of G joins two nodes of the same layer → G contains an odd-length cycle and hence is not bipartite
- Pf. (ii)
  - Suppose (x, y) is an edge with x, y in same level L_j.
  - Let z = lca(x, y) = lowest common ancestor
  - Let L_i be level containing z
  - Consider cycle that takes edge from x to y, then path y →z, then path from z → x



Analyzing Algorithm's Correctness

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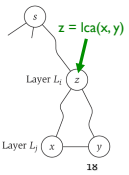
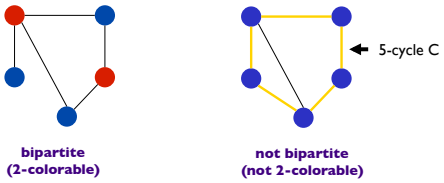
  - Let z = lca(x, y)=lowest common ancestor
  - Let L_i be level containing z
  - Consider cycle that takes edge from x to y, then path y → z, then path z → x
  - Its length is 1 + (j-i) + (j-i), which is odd
    - 1: (x, y)
    - (j-i): path from y to z
    - (j-i): path from z to x



An Obstruction to Bipartiteness

- Corollary. A graph G is bipartite *iff* it contains no odd length cycle.



bipartite (2-colorable)

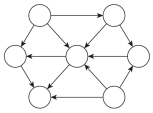
not bipartite (not 2-colorable)

Second verse, similar to the first.
But directed!

DIRECTED GRAPHS

Directed Graphs G = (V, E)

- Edge (u, v) goes from node u to node v



- Example: Web graph - hyperlink points from one web page to another
  - Directedness of graph is crucial
  - Modern web search engines exploit hyperlink structure to rank web pages by importance

Representing Directed Graphs

- For each node, keep track of
  - Out edges (where links go)
  - In edges (from where links come in)
  - Space required?
- Could only store *out* edges
  - Figure out *in* edges with increased computation/time
  - Useful to have both *in* and *out* edges

Rock Paper Scissors Lizard Spock

CONNECTIVITY IN DIRECTED GRAPHS

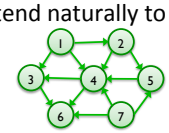
Graph Search

- How does ***reachability*** change with directed graphs?

- Example: Web crawler
  1. Start from web page s.
  2. Find all web pages linked from s, either directly or indirectly.

Graph Search

- Directed reachability. Given a node *s*, find all nodes reachable from *s*.
- Directed *s-t* shortest path problem. Given two nodes *s* and *t*, what is the length of the shortest path between *s* and *t*?
  - Not necessarily the same as t→s shortest path
- Graph search. BFS and DFS extend naturally to directed graphs
  - Trace through out edges
  - Run in O(m+n) time



Problem

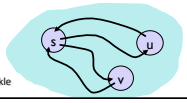
- Find all nodes with paths **to** *s*
  - Rather than paths from *s* to other nodes

Problem/Solution

- Problem. Find all nodes with paths **to** *s*
- Solution. Run BFS on *in edges* instead of out edges

Strong Connectivity

- Def. Node *u* and *v* are ***mutually reachable*** if there is a *path* from *u* → *v* and also a *path* from *v* → *u* (not necessarily a direct edge)
- Def. A graph is ***strongly connected*** if every pair of nodes is mutually reachable
- Lemma. Let *s* be any node. G is strongly connected ***iff*** every node is reachable from *s* and *s* is reachable from every node



Strong Connectivity

- If *u* and *v* are mutually reachable and *v* and *w* are mutually reachable, then *u* and *w* are mutually reachable

Strong Connectivity

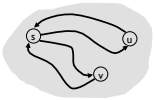
- If *u* and *v* are mutually reachable and *v* and *w* are mutually reachable, then *u* and *w* are mutually reachable.
- Proof. We need to show that there is a path from *u* → *w* and from *w* → *u*.
  - By defn of mutually reachable
    - There is a path *u* → *v* & a path *v* → *u*
    - There is a path *v* → *w*, and a path *w* → *v*
  - Take path *u*→*v* and then from *v* → w
    - Path from *u*→*w*
  - Similarly for *w*→*u*

Strong Connectivity

- Def. A graph is ***strongly connected*** if every pair of nodes is mutually reachable
- Lemma. Let *s* be any node. G is ***strongly connected*** *iff* every node is reachable from *s* and *s* is reachable from every node.
  - 1st prove ⇒
  - 2nd prove ⇐
    - for any nodes *u* and *v*, is there a path *u*→*v* and *v*→*u* ?

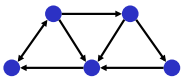
Strong Connectivity

- Def. A graph is strongly connected if every pair of nodes is mutually reachable
- Lemma. Let s be any node. G is ***strongly connected*** *iff* every node is reachable from s, and s is reachable from every node.
- Pf. ⇒ Follows from definition of strongly connected
- Pf. ⇐ For any nodes *u* and *v*, make path *u*→*v* and *v*→*u*
  - *u*→*v* : concatenating *u*→*s* with *s*→*v*
  - *v* →*u*: concatenate *v*→*s* with *s*→*u*

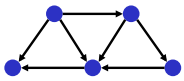


Strong Connectivity Problem

- Determine if G is strongly connected in O(m + n) time



strongly connected



not strongly connected

Hint: Can we leverage any algorithms we know have O(m+n) time?

Strong Connectivity: Algorithm

- Theorem. Can determine if G is strongly connected in O(m + n) time.
- Pf.
  - Pick any node *s*
  - Run BFS from *s* in G
  - Run BFS from *s* in G_{rev} (reverse orientation of every edge in G Or, the BFS using the *in* edges)
  - Return true *iff* **all** nodes reached in both BFS executions
  - Correctness follows immediately from previous lemma
    - All reachable from one node, *s* is reached by all

Strong Components

- For any two nodes *s* and *t* in a directed graph, their strong components are either identical or disjoint

Hint: Consider a node in common…

Strong Components

- For any two nodes *s* and *t* in a directed graph, their strong components are either identical or disjoint
- Proof.
  - Consider *v* in both strong components
    - s→ v; v → s; v→t; t→v ➔ t→s, s→t (mutually reachable)
    - As soon as there is one common node, then have identical strong components
  - On the other hand, consider *s* and *t* are not mutually reachable
    - No node *v* that is in the strong component of each
      - What would it mean if there were?

Looking Ahead

- Exam – due Wed, 5 p.m.
- See email about office hours
- No wiki for Monday