Objectives

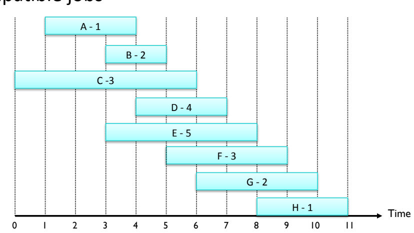
- Dynamic Programming
  - Wrapping up: weighted interval schedule
  - Segmented Least Squares
  - Subset Sums

Summary: Properties of Problems for DP

- Polynomial number of subproblems
- Solution to original problem can be easily computed from solutions to subproblems
- Natural ordering of subproblems, easy to compute recurrence

Review: Weighted Interval Scheduling

- Job j starts at s_j, finishes at f_j, and has weight or value v_j
- Two jobs are compatible if they don't overlap
- **Goal**: find maximum weight subset of mutually compatible jobs



Weighted Interval Scheduling: Memoization Analysis

```
Input: n jobs (associated start time sj, finish time fj, and
value vj)

Sort jobs by finish times so that f1 ≤ f2 ≤ ... ≤ fn    O(n log n)
Compute p(1), p(2), …, p(n)   O(n log n)

for j = 1 to n
   M[j] = empty     O(n)
M[0] = 0

M-Compute-Opt(j):
   if M[j] is empty:
      M[j] = max(vj + M-Compute-Opt(p(j)), M-Compute-Opt(j-1))
   return M[j]

M-Compute-Opt(n)  O(n)
```


Weighted Interval Scheduling: Finding a Solution

- Dynamic programming algorithms compute ***optimal value***
- What if we want the ***solution*** itself (**not** simply the value)?
- Do some post-processing

```
M-Compute-Opt(n)
Find-Solution(n)

def Find-Solution(j):
   if j = 0:
      output nothing
   elif vj + M[p(j)] > M[j-1]:
      print j
      Find-Solution(p(j))
   else:
      Find-Solution(j-1)
```

Runtime?

O(n)

Turning it Around…

- We solved the Fibonacci problem as both recursive/memoized and an **iterative** algorithm

Can we write this algorithm as an **iterative** solution?

```
Input: n jobs (associated start time sj, finish time fj, and
value vj)

Sort jobs by finish times so that f1 ≤ f2 ≤ ... ≤ fn
Compute p(1), p(2), …, p(n)

for j = 1 to n
   M[j] = empty
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M-Compute-Opt(j):
   if M[j] is empty:
      M[j] = max(vj + M-Compute-Opt(p(j)), M-Compute-Opt(j-1))
   return M[j]
M-Compute-Opt(n)
```
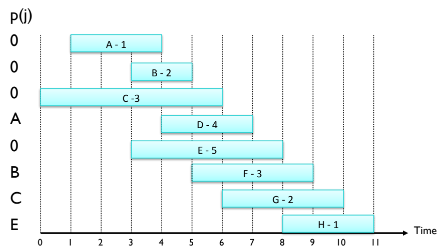

Towards Iterative Solution…



M

0	A	B	C	D	E	F	G	H

Iterative Solution

- Build up solution from subproblems instead of breaking down

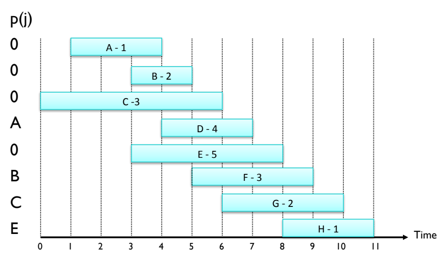
```
Input: n, s_1,…,s_n , f_1,…,f_n , v_1,…,v_n

Sort jobs by finish times so that f_1 ≤ f_2 ≤ ... ≤ f_n.

Compute p(1), p(2), …, p(n)

M[0] = 0
for j = 1 to n
    M[j] = max(v_j + M[p(j)], M[j-1])
```

Runtime?

O(n)

- Typically, we'll take iterative approach

Example: Iteratively

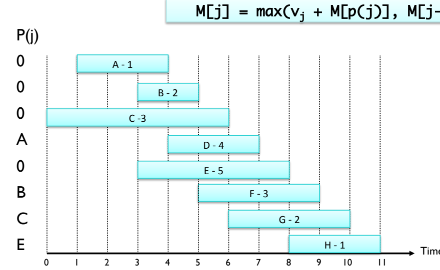


M

0	A	B	C	D	E	F	G	H
0								

Example: Iteratively

M[j] = max(v_j + M[p(j)], M[j-1])

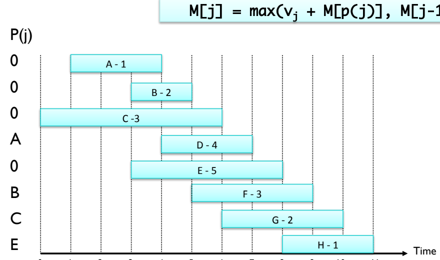


M

0	A	B	C	D	E	F	G	H
0								

Example: Iteratively

M[j] = max(v_j + M[p(j)], M[j-1])

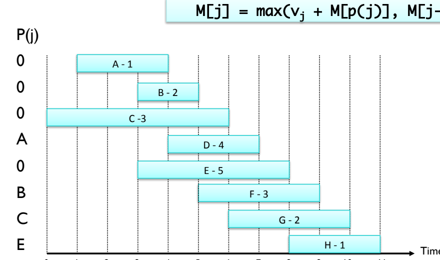


M

0	A	B	C	D	E	F	G	H
0	1							

Example: Iteratively

M[j] = max(v_j + M[p(j)], M[j-1])

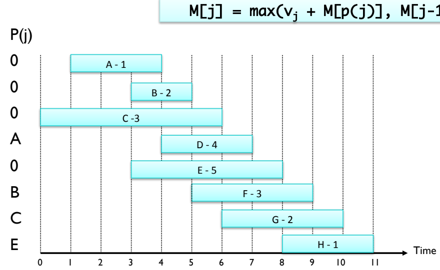


M

0	A	B	C	D	E	F	G	H
0	1							

Example: Iteratively

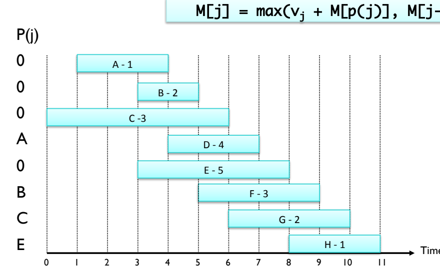
$M[j] = \max(v_j + M[p(j)], M[j-1])$



M	0	A	B	C	D	E	F	G	H
	0	**1**	**2**						

Example: Iteratively

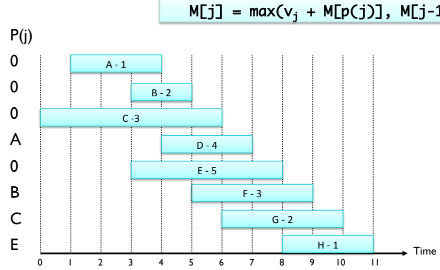
$M[j] = \max(v_j + M[p(j)], M[j-1])$



M	0	A	B	C	D	E	F	G	H
	0	**1**	**2**	**3**					

Example: Iteratively

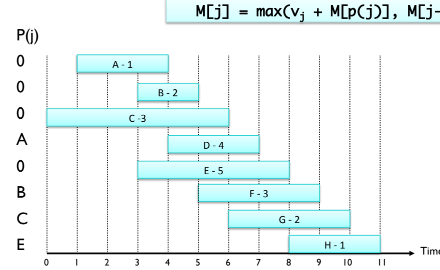
$M[j] = \max(v_j + M[p(j)], M[j-1])$



M	0	A	B	C	D	E	F	G	H
	0	**1**	**2**	**3**	**5**				

Example: Iteratively

$M[j] = \max(v_j + M[p(j)], M[j-1])$

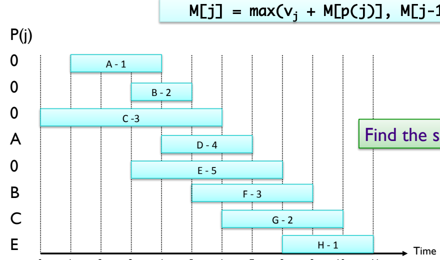


M	0	A	B	C	D	E	F	G	H
	0	**1**	**2**	**3**	**5**				

And so on….

Example: Iteratively

$M[j] = \max(v_j + M[p(j)], M[j-1])$



Find the solution?

M	0	A	B	C	D	E	F	G	H
	0	**1**	**2**	**3**	**5**	**5**	**5**	**5**	**6**

Review: Weighted Interval Scheduling

1. Determine optimal substructure of problem
   - Define the recurrence relation
2. Define algorithm to find the ***value*** of optimal solution
3. Optionally, change algorithm to an ***iterative*** rather than recursive solution
4. Define algorithm to find ***optimal solution***
5. Analyze running time of algorithms

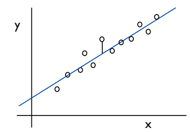
Map to weighted-interval scheduling

SEGMENTED LEAST SQUARES

Least Squares

- Foundational problem in statistic and numerical analysis
- Given *n* points in the plane: (x_1, y_1), (x_2, y_2) , . . . , (x_n, y_n)
- Find a line $y = ax + b$ that minimizes the sum of the squared error
  - "line of best fit"

Sum of squared error $SSE = \sum_{i=1}^{n} (y_i - ax_i - b)^2$



Least Squares

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- Given *n* points in the plane: (x_1, y_1), (x_2, y_2) , . . . , (x_n, y_n)
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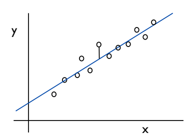
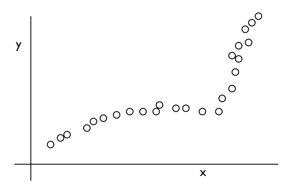


Sum of squared error $SSE = \sum_{i=1}^{n} (y_i - ax_i - b)^2$

- Closed form solution. Calculus $\Rightarrow$ min error is achieved when $a = \frac{n \sum_i x_i y_i - (\sum_i x_i)(\sum_i y_i)}{n \sum_i x_i^2 - (\sum_i x_i)^2}, \quad b = \frac{\sum_i y_i - a \sum_i x_i}{n}$

Least Squares

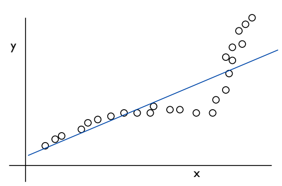
- What happens to the error if we try to fit one line to these points?



- What pattern does it seem like these points have?

Least Squares

- What happens to the error if we try to fit one line to these points?
  - Large error

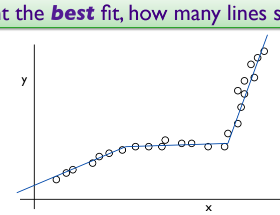


- Pattern: More like 3 lines

Segmented Least Squares

- Points lie roughly on a **sequence** of line segments
- Given *n* points in the plane (x_1, y_1), (x_2, y_2) , . . . , (x_n, y_n) with $x_1 < x_2 < ... < x_n$, find a **sequence** of **line segments** that **minimizes** **f(x)**

If I want the ***best*** fit, how many lines should I use?



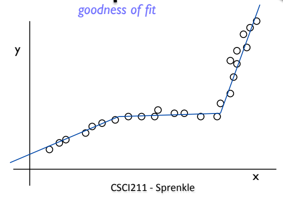
Segmented Least Squares

- Points lie roughly on a **sequence** of line segments
- Given *n* points in the plane $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ with $x_1 < x_2 < \ldots < x_n$, find a sequence of line segments that **minimizes f(x)**

What's a reasonable choice for *f(x)* to balance *accuracy* and *parsimony*?

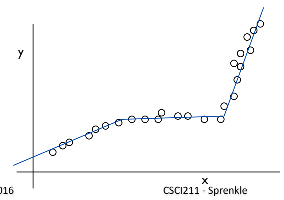
goodness of fit

number of lines



Segmented Least Squares

- Points lie roughly on a **sequence** of several line segments.
- Given *n* points in the plane $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ with $x_1 < x_2 < \ldots < x_n$, find a sequence of line segments that minimizes:
  - *E*: sum of the sums of the squared errors in each segment
  - *L*: the number of lines
- **Tradeoff function**: $E + c\,L$, for some constant $c > 0$.



How should we define an optimal solution?

Segmented Least Squares

- What made it seem like the points were in 3 lines? What happened?

Segmented Least Squares

- What made it seem like the points were in 3 lines? What happened?

- Error increased
- Looking for *change* in linear approximation
  - Where to partition points into line segments

Recall: Properties of Problems for DP

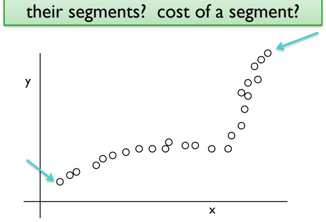
- Polynomial number of subproblems
- Solution to original problem can be easily computed from solutions to subproblems
- Natural ordering of subproblems, easy to compute recurrence

We need to:
- Figure out how to break the problem into subproblems
- Figure out how to compute solution from subproblems
- Define the recurrence relation between the problems

Toward a Solution

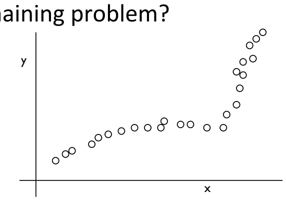
- Consider just the first or last point

What do we know about those points? their segments? cost of a segment?



Toward a Solution

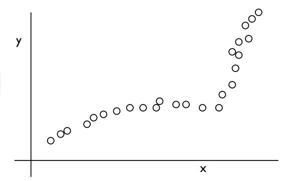
- p_n can only belong to one segment
  - Segment: p_i, …, p_n
  - Cost: c (cost for segment) + error of segment
- What is the remaining problem?



Toward a Solution

- p_n can only belong to one segment
  - Segment: p_i, …, p_n
  - Cost: c (cost for segment) + error of segment
- What is the remaining problem?
  - Solve for p_1, …, p_{i-1}

Next: Formulate as a recurrence



Dynamic Programming: Multiway Choice

- Notation.
  - **OPT(j)** = minimum cost for points $p_1, p_{i+1}, \ldots, p_j$.
  - **e(i, j)** = minimum sum of squares for points $p_i, p_{i+1}, \ldots, p_j$.
- How do we compute OPT(j)?
  - Last problem: binary decision (include job or not)
  - This time: **multiway** decision
    - Which option do we choose?

Dynamic Programming: Multiway Choice

- Notation.
  - **OPT(j)** = minimum cost for points $p_1, p_{i+1}, \ldots, p_j$.
  - **e(i, j)** = minimum sum of squares for points $p_i, p_{i+1}, \ldots, p_j$.
- To compute OPT(j):
  - Last segment contains points $p_i, p_{i+1}, \ldots, p_j$ for some i
  - Cost = e(i, j) + c + OPT(i-1).

$$OPT(j) = \begin{cases} 0 & \text{if } j = 0 \\ \min_{1 \le i \le j} \{ e(i,j) + c + OPT(i-1) \} & \text{otherwise} \end{cases}$$

Segmented Least Squares: Algorithm

```
INPUT: n, p1,…,pN , c

Segmented-Least-Squares()
   M[0] = 0
   e[0][0] = 0  # needed?
   for j = 1 to n
      for i = 1 to j
         e[i][j] = least square error for the
                      segment pi, …, pj

   for j = 1 to n
      M[j] = min 1 ≤ i ≤ j (e[i][j] + c + M[i-1])

   return M[n]
```

Costs?

Segmented Least Squares: Algorithm Analysis

How do we find the *solution?*

can be improved to $O(n^2)$ by pre-computing various statistics

$O(n^3)$

```
INPUT: n, p1,…,pN , c

Segmented-Least-Squares()
   M[0] = 0
   e[0][0] = 0
   for j = 1 to n
      for i = 1 to j
         e[i][j] = least square error for the
                      segment pi,…, pj

   for j = 1 to n
      M[j] = min 1 ≤ i ≤ j (e[i][j] + c + M[i-1])

   return M[n]
```

$O(n^2)$

- Bottleneck: computing e(i, j) for $O(n^2)$ pairs, O(n) per pair using previous formula

Post-Processing: Finding the Solution

```
FindSegments(j):
   if j = 0:
       output nothing
   else:
       Find an i that minimizes e_{i,j} + c + M[i-1]
       Output the segment {p_i, …, p_j}
       FindSegments(i-1)
```

Cost? O(n^2)

Dynamic Programming Process

- Determine optimal substructure of problem
  - Define the recurrence relation
- Define algorithm to find the ***value*** of optimal solution
- Optionally, change algorithm to an ***iterative*** rather than recursive solution
- Define algorithm to find ***optimal solution***
- Analyze running time of algorithms

Looking Ahead

- Wiki – Monday
  - Sections 6-6.3
- Problem Set 8 – due Friday