Objectives

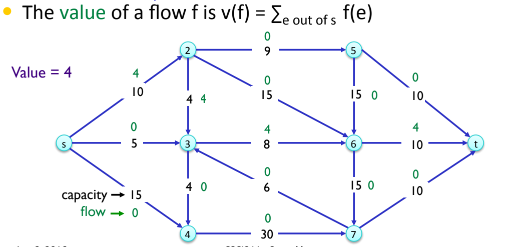
- Network Flow
  - Circulation
  - Application: Survey Design
  - Application: Airline Scheduling

Review

- What is a flow network?
- What is our usual goal given a flow network?
  - How do we reach that goal?
- What is the Ford-Fulkerson algorithm?
- What is the min-cut?
  - How does it relate to the max flow?

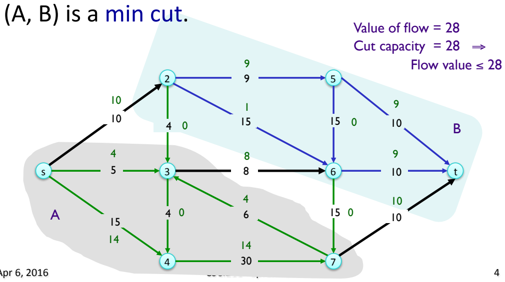
Review: Network Flows

- An **s-t flow** is a function that satisfies
  - ***Capacity* condition**: For each $e \in E$: $0 \le f(e) \le c(e)$
  - ***Conservation* condition**: For each $v \in V - \{s, t\}$: $\sum_{e \text{ into } y} f(e) = \sum_{e \text{ out of } y} f(e)$
- The value of a flow f is $v(f) = \sum_{e \text{ out of } s} f(e)$



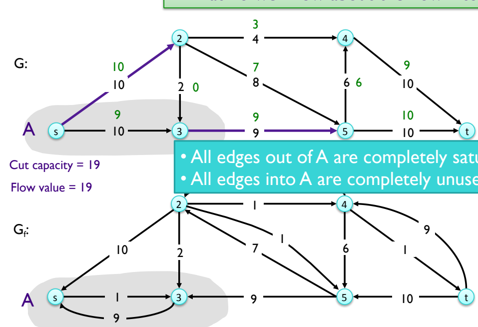
Review: Certificate of Optimality

- Corollary. Let *f* be any flow, and let (A, B) be any cut. If v(f) = cap(A, B), then *f* is a max flow and (A, B) is a min cut.



Review: Ford-

- What do we know about the flow out of A?
- What do we know about the flow into A?



- All edges out of A are completely saturated
- All edges into A are completely unused

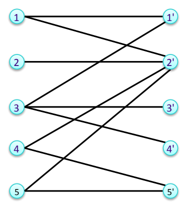
Power of Max Flow Problem

Some problems with non-trivial combinatorial searches can be formulated as **max flow** or **min cut** in a directed graph

Review: Bipartite Graph: Max Flow Formulation

Problem: find matching of largest possible size

How did we turn this into a max flow problem?

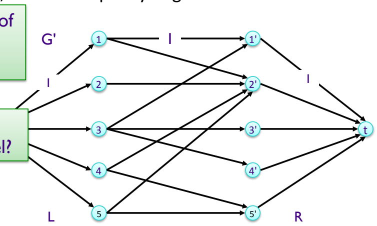


Review: Bipartite Graph: Max Flow Formulation

- Create digraph G' = (L ∪ R ∪ {s, t}, E')
- Direct all edges from L to R, and assign unit capacity
- Add source s, and unit capacity edges from s to each node in L
- Add sink t, and unit capacity edges from each node in R to t

What is cost of generating model?

What is C in this model?



Summary of Approach

1. Model problem as a flow network
2. Run Ford-Fulkerson algorithm
   - Map back to original problem
3. Prove that the solution found is correct/ feasible/optimal
4. Prove that you find all solutions
5. Analyze running time
   1. Creating model
   2. FF algorithm

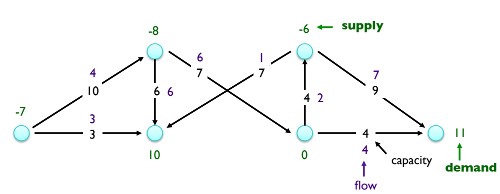
Section 7.7

EXTENSIONS TO MAX FLOW

Circulation with Demands

- Directed graph G = (V, E)
- Edge capacities c(e), e ∈ E
- Node supply and demands d(v), v ∈ V
  - d(v) > 0 → demand
  - d(v) < 0 → supply
  - d(v) = 0 → transshipment

Example Graph: Circulation with Demands



- d(v) > 0 → demand
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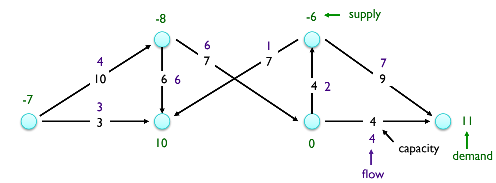
Circulation with Demands

- Circulation with demands
  - Directed graph G = (V, E)
  - Edge capacities c(e), e ∈ E
  - Node supply and demands d(v), v ∈ V
- Def. A ***circulation*** is a function that satisfies:
  - For each e ∈ E: $0 \le f(e) \le c(e)$ (capacity)
  - For each v ∈ V: $\sum_{e \text{ in to } v} f(e) - \sum_{e \text{ out of } v} f(e) = d(v)$ (conservation)

Circulation problem:
given (V, E, c, d), does a circulation exist?
(Can we satisfy demand with supply?)

Example Graph: Circulation with Demands

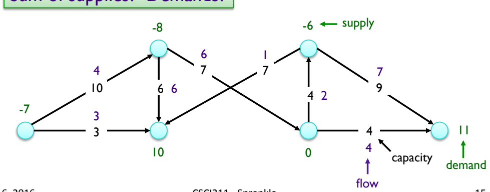


Circulation with Demands

- Necessary condition:
  sum of supplies = sum of demands

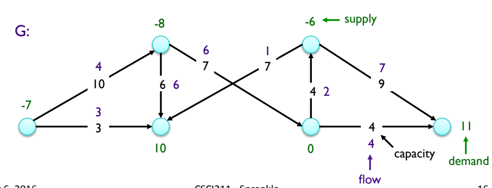
$$\sum_{v:d(v)>0} d(v) = \sum_{v:d(v)<0} -d(v) =: D$$

Sum of supplies? Demands?



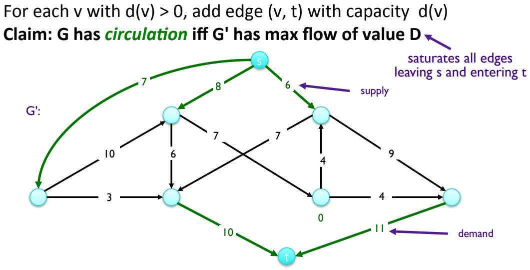
Circulation with Demands: Towards Max Flow Formulation

Ideas about how we can formulate this as a max flow problem?



Circulation with Demands: Max Flow Formulation

- Add new source s and sink t
- For each v with d(v) < 0, add edge (s, v) with capacity -d(v)
- For each v with d(v) > 0, add edge (v, t) with capacity d(v)
- **Claim: G has *circulation* iff G' has max flow of value D**



Circulation with Demands: Characterization

- Given (V, E, c, d), there does **not** exist a circulation iff there exists a node partition (A, B) such that

$$\Sigma_{v \in B} d_v > cap(A, B)$$

demand by nodes in B — exceeds — supply of nodes in B + max capacity of edges going from A → B

- Proof?
  - What can we use to prove this?

Circulation with Demands: Characterization

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- Pf idea. Look at min cut in G'.

ANOTHER EXTENSION: LOWER BOUNDS

Circulation with Demands and Lower Bounds

Force flow to use certain edges

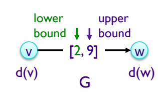
- Feasible circulation
  - Directed graph G = (V, E)
  - Edge capacities c(e) and lower bounds ℓ (e), e ∈ E
  - Node supply and demands d(v), v ∈ V

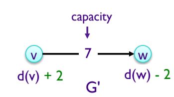
- Def. A *circulation* is a function that satisfies:
  - For each e ∈ E: 0 ≤ ℓ (e) ≤ f(e) ≤ c(e) (capacity)
  - For each v ∈ V: $\sum_{e \text{ in to } v} f(e) - \sum_{e \text{ out of } v} f(e) = d(v)$ (conservation)

Circulation problem with lower bounds.
Given (V, E, ℓ, c, d), does a circulation exist?

Circulation with Demands and Lower Bounds

- Model lower bounds with demands
  - Send ℓ(e) units of flow along edge e
  - Update demands of both endpoints





Supply and demand decrease

Proof in book

Circulation with Demands and Lower Bounds

Force flow to use certain edges

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Circulation problem with lower bounds.
Given (V, E, ℓ, c, d), does a circulation exist?

7.8 SURVEY DESIGN

Survey Design

- Design survey asking consumers about products
- Can only survey a consumer about a product if they own it
  - Consumer can own multiple products
- Ask consumer i between c_i and c_i' questions
- Ask between p_j and p_j' consumers about product j

Goal: Design a survey that meets these specs, if possible.

How can we model this problem?

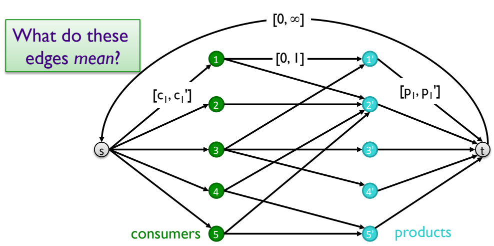
Model: Bipartite Graph

- Nodes: customers and products
- Edge between customer and product means customer owns product
- For each customer, range of # of products asked about
- For each product, range of # of customers asked about it

What does the flow represent?

Survey Design Algorithm

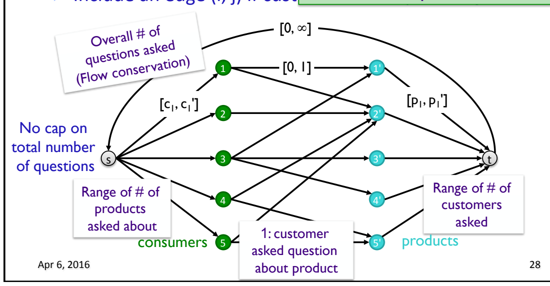
- Formulate as a circulation problem with lower bounds
  - Include an edge (i, j) if customer i owns product j



Survey Design Algorit

Alternative bounds on t→s?
How do we know if we can create a survey?
What is the survey?
How many solutions are there to this problem?

- Formulate as a circulation bounds
  - Include an edge (i, j) if cust



Survey Solution

- If a feasible, integer flow solution, can create the survey
- Customer *i* will be surveyed about product *j* iff the edge (*i*,*j*) carries a unit of flow

Looking Ahead

- Problem Set 9 – due Friday
- Course Evaluations, due Monday
  - on Sakai – under "Tests and Quizzes"
  - Up to 5% added to your problem set score
  - If 60% of students complete, 1% added to problem set
  - For each additional 10% of class that completes survey, additional 1% bonus added to problem set