# **Objectives**

- Analyzing proofs
- Introduction to problem solving
  - > Our process, through an example

4 p.m. – Alicia Grubb, faculty candidate talk, P405 3:30 p.m. reception

#### Wiki:

- Everyone log in okay?
- Decide on either using a blog or wiki-style journal?

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#### Review

- What are our goals in solving problems?
- How do we show that our solutions are correct and efficient?
- What proof techniques did we discuss?

# **Proof Summary**

- Need to *prove* conjectures
- Common types of proofs
  - Direct proofs
  - **Contradiction**
  - **►** Induction
- Common error: not checking/proving assumptions
  - "Jumps" in logic

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Process, through example

# INTRODUCTION TO PROBLEM SOLVING

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#### Matching Residents to Hospitals

- Goal: Given a set of preferences among hospitals and medical school students, design a self-reinforcing admissions process.
- Applicant a and hospital h are unstable if
  - a prefers h to its assigned hospital
  - $\triangleright$  h prefers  $\alpha$  to one of its admitted students
- **Stable assignment**: Assignment with no unstable pairs
  - No incentive for some pair of participants to undermine assignment by joint action
    - Unstable pair could each improve their situation by swapping with current assignment

What details make this problem tricky?

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What info do we need to solve problem?

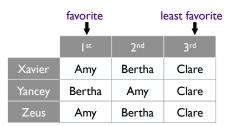
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#### Stable Matching Problem

Simplified version of resident-matching problem

- **Goal**: Given *n* men and *n* women, find a "suitable" matching
  - Participants rank members of opposite sex
  - Each man ranks women in order of preference
  - Each woman ranks men in order of preference



favorite least favorite Yancey **Xavier** Zeus Xavier Zeus Bertha Yancey Clare Xavier Yancey Zeus

Men's Preference Profile

Women's Preference Profile

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#### **Stable Matching Goals**

- Perfect matching: everyone is matched monogamously
  - Each man is paired with exactly one woman
  - Each woman is paired with exactly one man
- Stability: no incentive for some pair of participants to undermine assignment by joint action
  - An unmatched pair m-w is unstable if man m and woman w prefer each other to current partners
  - Unstable pair m-w could each improve by eloping
- Stable matching: perfect matching with no unstable pairs

#### Stable matching problem:

Given the preference lists of n men and n women, find a stable matching if one exists.

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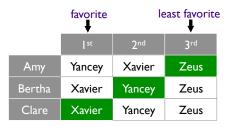
#### **Analyzing Stability**

**Instable**: *m* prefers *w* to his woman; *w* prefers *m* to her man

Is pairing X-C, Y-B, Z-A stable?



Men's Preference Profile



Women's Preference Profile

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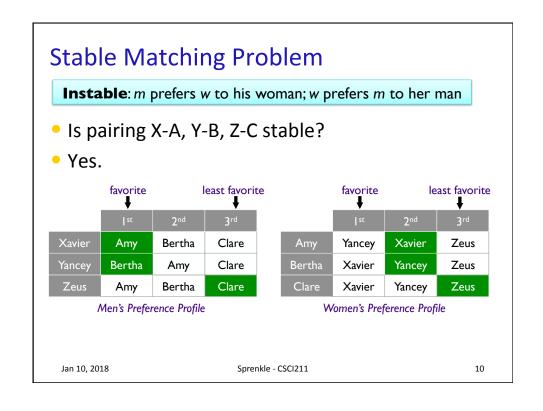
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#### **Analyzing Stability** Is pairing X-C, Y-B, Z-A stable? No. Bertha and Xavier prefer each other favorite least favorite favorite least favorite Amy Bertha Clare Yancey **Xavier** Zeus Bertha Amy Clare Bertha Xavier Yancey Zeus Bertha Clare Zeus Amy Clare Xavier Yancey Zeus Men's Preference Profile Women's Preference Profile

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# **Any Questions?**

 What are you wondering about this problem/its solution at this point?

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# **Any Questions?**

- What are you wondering about this problem/its solution at this point?
- Hopefully:
  - ➤ Is there a stable matching for every pair of preference lists?
  - ➢ If so, is there an algorithm to find the stable matching?
  - > Can we be fair in the matching? (preferences)
  - Will the matching always be the same?

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### **Thoughts on Solving Problem**

- What do we need to solve the problem?
- What do we know?
- Where should the state start?
- What are some initial ideas about approaches?

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#### **Thoughts on Solving Problem**

- Initially, no one is matched
- Pick an arbitrary man and have him match with his favorite woman.
  - Are we guaranteed that pair will be part of a stable matching?
- Should a woman accept her first offer? If not, what should she do?
- When are we done? Do we need to consider all combinations?

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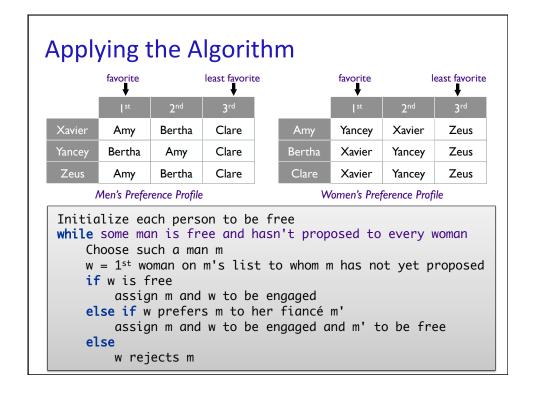
#### Propose-And-Reject Algorithm

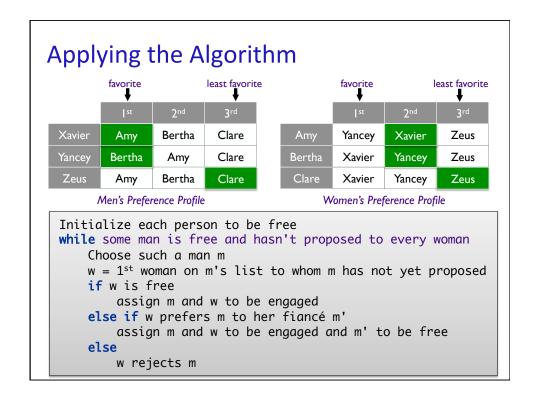
[Gale-Shapley 1962]

 Intuitive method that guarantees finding a stable matching

```
Initialize each person to be free
while some man is free and hasn't proposed to every woman
Choose such a man m
w = 1st woman on m's list to whom m has not yet proposed
if w is free
assign m and w to be engaged
else if w prefers m to her fiancé m'
assign m and w to be engaged and m' to be free
else
w rejects m
```

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#### Observations about the Algorithm

- What can we say about any woman's partner during the execution of the algorithm?
- How does a woman's state change over the execution of the algorithm?
- What can we say about a man's partner?

#### Observations about the Algorithm

- What can we say about any woman's partner during the execution of the algorithm?
  - ➤ Observation 1. He gets "better" → she prefers him over her last partner
- How does a woman's state change over the execution of the algorithm?
  - ➤ Observation 2. Once a woman is matched, she never becomes unmatched; she only "trades up"
- What can we say about a man's partner?
  - Observation 3. She gets "worse"

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#### **Proving Correctness**

- Need to show
  - > Algorithm terminates



Result is a stable matching



# 1) Algorithm Termination

[Gale-Shapley 1962]

#### Does algorithm terminate?

```
Initialize each person to be free
while (some man is free and hasn't proposed to every woman)
    Choose such a man m
    w = 1<sup>st</sup> woman on m's list to whom m has not yet proposed
    if w is free
        assign m and w to be engaged
    else if w prefers m to her fiancé m'
        assign m and w to be engaged and m' to be free
    else
        w rejects m
```

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#### **Proof of Correctness: Termination**

- Claim. Algorithm terminates after at most n<sup>2</sup> iterations of while loop.
  - Hint: How wouldn't the algorithm terminate?

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#### **Proof of Correctness: Termination**

- Claim. Algorithm terminates after at most n<sup>2</sup> iterations of while loop.
- Pf. Each time through the while loop, a man proposes to a new woman. There are only n<sup>2</sup> possible proposals.

Number of proposals is a good measure for termination → strictly increases; limited

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#### **Proof of Correctness: Termination**

- Claim. Algorithm terminates after at most n<sup>2</sup> iterations of while loop.
- Pf. Each time through the while loop, a man proposes to a new woman. There are only n<sup>2</sup> possible proposals.

Note: not yet discussing the cost in the body of the while loop

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#### 2) Algorithm Analysis: Perfect Matching

#### Prove that final matching is a perfect matching

- Perfect matching: everyone is matched monogamously
- Hint: in algorithm, we know if m is free at some point in the execution of the algorithm, then there is a woman to whom he has not yet proposed.

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#### **Proof of Correctness: Perfection**

- Claim. All men and women get matched.
- Pf. (by contradiction)
  - > Where should we start?

Suppose that some man m is not matched upon termination of algorithm

#### **Proof of Correctness: Perfection**

- Claim. All men and women get matched.
- Pf. (by contradiction)
  - Suppose that m is not matched upon termination of algorithm
  - Then some woman, say w, is not matched upon termination.
  - > By Observation 2, w was never proposed to.
  - But, last man proposed to everyone, since he ends up unmatched

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- (by the while loop's condition)
- Contradiction •

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#### **Assignments**

- Review Chapter 1
- Journal due Monday/Tuesday (because of MLK day)
  - ➤ Preface, Chapter 1.1
  - Check out the content requirements for the journal entries

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