Objectives

- Review implementation of Stable Matching
- Survey of common running times

Review: Asymptotic Analysis of Gale-Shapley Alg

Not explicitly in the algorithm, but we need to make the inverse array before the while loop too.

```
Initialize each person to be free    O(n)
while (some man is free and hasn't proposed to every woman)    O(n^2)
    Choose such a man m    O(1)
    w = 1st woman on m's list to whom m has not yet proposed    O(1)
    if (w is free)    O(1)
        assign m and w to be engaged    O(1)
    else if (w prefers m to her fiancé m')    O(1)  Using inverse array
        assign m and w to be engaged and m' to be free    O(1)
    else
        w rejects m    O(1)
```

Total: $O(n^2)$

More Explicit Algorithm - Preferred

```
def stableMatching( men, women, men_pref_array,
                    women_pref_array):
O(n)  Initialize each person to be free (set up data structures)
O(n²) Create inverse array for women's preferences            O(n²)
   while (some man is free and hasn't proposed to every woman)
      Choose such a man m O(1)                          O(1)
      w = 1st woman on m's list to whom m has not yet proposed
      if (w is free) O(1)
         assign m and w to be engaged O(1)              Using inverse
      else if (w prefers m to her fiancé m') O(1)            array
         assign m and w to be engaged and m' to be free O(1)
      else
         w rejects m O(1)
   return engagements
```

Total: $O(n^2)$

A SURVEY OF COMMON RUNNING TIMES

Linear Time: O(n)

- Running time is at most a ***constant*** factor times the size of the input
- Example. Computing the maximum: Compute maximum of n numbers $a_1, \ldots, a_n$

```
max = a1
for i = 2 to n
    if (ai > max)
        max = ai
```

Constant work for each input (does not depend on n)

Example Linear Time: O(n)

- Merge: Combine two sorted lists $A = a_1, a_2, \ldots, a_n$ with $B = b_1, b_2, \ldots, b_n$ into sorted whole

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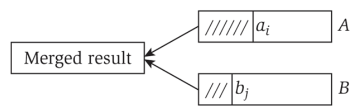
- Merge: Combine two sorted lists $A = a_1, a_2, \ldots, a_n$ with $B = b_1, b_2, \ldots, b_n$ into sorted whole
- Claim. Merging two lists of size *n* takes O(n) time

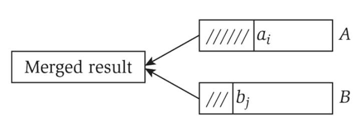



```
i = 1, j = 1
while (both lists are nonempty)
   if (ai ≤ bj)
        append ai to output list and increment i
   else (ai ≤ bj)
        append bj to output list and increment j

append remainder of nonempty list to output list
```


Example Linear Time: O(n)

- Merge: Combine two sorted lists $A = a_1, a_2, \ldots, a_n$ with $B = b_1, b_2, \ldots, b_n$ into sorted whole
- Claim. Merging two lists of size n takes O(n) time
- Proof. After each comparison, the length of output list increases by 1

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O(n log n) Time

- Also referred to as ***linearithmic*** time
- Arises in divide-and-conquer algorithms
  - Splitting input into equal pieces, solve recursively, combine solutions in linear time

What well-known set of algorithms has an O(n logn) running time?

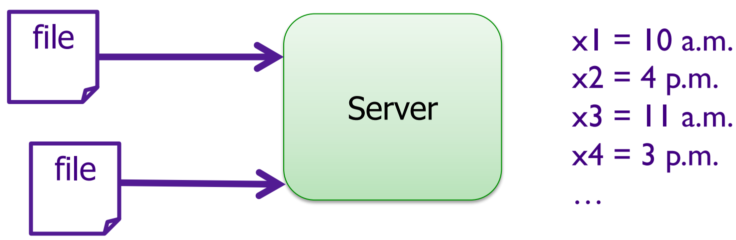
O(n log n) Time Example

- Sorting: Mergesort and heapsort are sorting algorithms that perform O(n log n) comparisons
- Mergesort
  1. Break input into equal-sized pieces
  2. Sorts each half recursively
  3. Merges sorted halves into a sorted list

Talk about the bound on running time later…

O(n log n) Time Example

- Largest empty interval. Given *n* (not necessarily ordered) time-stamps $x_1, \ldots, x_n$ at which copies of a file arrive at a server, what is largest interval of time when no copies of the file arrive?



O(n log n) Time Example

- Largest empty interval. Given *n* (not necessarily ordered) time-stamps $x_1, \ldots, x_n$ at which copies of a file arrive at a server, what is largest interval of time when no copies of the file arrive?
- O(n log n) solution
  1. Sort time-stamps
  2. Scan sorted list in order, identifying the maximum gap between successive time-stamps

Quadratic Time: $O(n^2)$

- Examples?

Quadratic Time: $O(n^2)$

- Examples:
  - Enumerate all pairs of elements
  - Sometimes involves nested loops (n iterations)

Quadratic Time: $O(n^2)$

- Closest pair of points. Given a list of n points in the plane (x_1, y_1), …, (x_n, y_n), find the pair that is closest
- $O(n^2)$ solution. Try all pairs of points

```
min = (x1 - x2)^2 + (y1 - y2)^2
for i = 1 to n
   for j = i+1 to n
      d = (xi - xj)^2 + (yi - yj)^2
      if (d < min)
         min = d
```

don't need to take square roots

$\Omega(n^2)$ seems inevitable, but Chapter 5 has an O(n logn) solution

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Polynomial Time: $O(n^k)$ Time

- To get all pairs, the algorithm is $O(n^2)$
- To get all triplets, the algorithm is $O(n^3)$

What is an example of an $O(n^k)$ algorithm?

All subsets of size *k*

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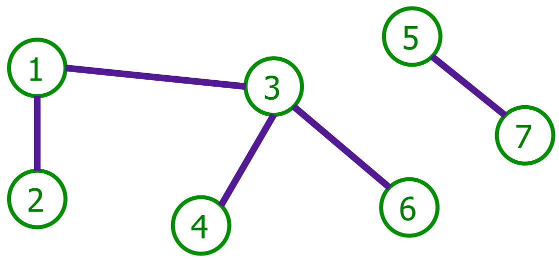
Polynomial Time: O(n^k) Time

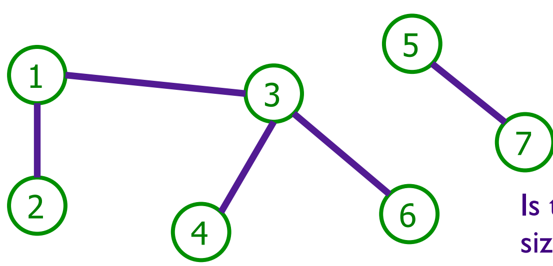
- Independent set of size *k*. Given a graph, are there *k* nodes such that no two are joined by an edge?
  - *k* is a constant



Is there an independent set of size 2? 3? 4? 5?

Polynomial Time: O(n^k) Time

- Independent set of size *k*. Given a graph, are there *k* nodes such that no two are joined by an edge?
  - *k* is a constant



Is there an independent set of size 2? Yes (2-3; 1-5; 6-7; …)
3? (5-6-7; 2-3-5; …)
4? (2-4-6-7; 1-4-6-7; …)
But not 5

Polynomial Time: O(n^k) Time

- Independent set of size *k*. Given a graph, are there *k* nodes such that no two are joined by an edge?
  - *k* is a constant

```
foreach subset S of k nodes
   check whether S in an independent set
   if (S is an independent set)
      report S is an independent set
```

- O(n^k) solution
  1. Enumerate all subsets of k nodes
     $$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{n!}{k!\,(n-k)!} = \frac{n\,(n-1)\,(n-2)\ldots(n-k+1)}{k\,(k-1)\,(k-2)\ldots(2)\,(1)} \leq \frac{n^k}{k!}$$
  2. Check whether S is an independent set = O(k^2).

$O(k^2\, n^k / k!) = O(n^k)$

poly-time for k=17 but not practical

Exponential Time

- Independent set. Given a graph, what is the *maximum size* of an independent set?
- O($n^2\, 2^n$) solution. Enumerate all subsets

```
S* = ϕ
foreach subset S of nodes
   check whether S in an independent set
   if (S is largest independent set seen so far)
      S* = S
```

O(log n) Time

- ***Sublinear*** time
- Know any algorithms that take O(log n) time?

O(log n) Time

- Example: Binary search
- Often requires some pre-processing or data structure that allows cheaper "querying" than *n* time

Summary of Running Times

Running Time	Example
O(log n)	Generally dividing problem in half on each iteration
O(n)	Operate on each input value
O(n log n)	Divide and conquer
O(n^2)	Operate on each pair of inputs
O(n!)	Operate on each permutation of inputs

MORE COMPLEX DATA STRUCTURES

Improving Running Times

After overcoming higher-level obstacles, lower-level **implementation details** can **improve runtime**.

PRIORITY QUEUES

Priority Queues

- Elements have a ***priority*** or *key*
- Each time select an element from the priority queue, want the one with *highest* priority
- More formally…
  - Maintains a set of elements S
    - Each element $v \in S$ has a $\mathrm{key}(v)$ for its priority
      - Smaller keys represent higher priorities
  - Application Programming Interface
    - Add, delete elements
    - Select element with smallest key

Key	**2**	**4**	**5**	**6**	**9**	**20**
Value	3542	5143	8712	1264	9123	5954

Priority (Key) ← Priority
Value ← Process id

(Not implementation, just how to envision)

Motivating Example: Scheduling Processes

Key	**2**	**4**	**5**	**6**	**9**	**20**
Value	3542	5143	8712	1264	9123	5954

Key ← Priority
Value ← Process id

- Each process has a priority or urgency
- Processes do not arrive in priority order
- **Goal**: run process with highest priority

Using a Priority Queue (PQ)

- PQ API:
  - Add an element with a given key (i.e., priority)
  - Delete an element with a given priority
  - Select element with smallest key/highest priority

Given a list of numbers, how could you use a PQ to sort that list of numbers?

Priority Queues for Sorting

1. Add elements into PQ with the number's value as its priority
2. Then extract the smallest number *until* done
   - Come out in sorted order

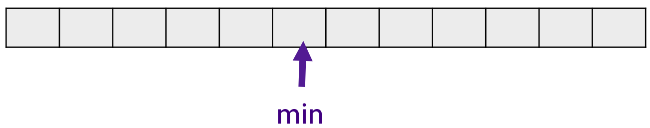
Sorting *n* numbers takes O(n logn) time

What is the goal running time for our PQ's operations? **O(logn)**

Already know our "loops" will be O(n)

Implementing a Priority Queue

- Consider an *unordered* list, where there is a pointer to minimum



- How difficult (i.e., expensive) is
  - Adding new elements?
  - Extraction?

Looking Ahead

- Wiki tonight – 2.3
- Problem Set 2 due Friday