Objectives

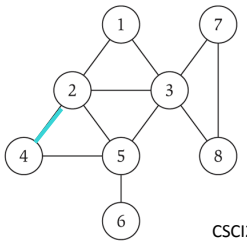
- Graphs
- Graph Connectivity, Traversal
- BFS & DFS Implementations, Analysis

Review

- What is a heap?
  - When is it useful?
- What is a graph?
  - What is one way to implement a graph?

Graph Representation: Adjacency Matrix

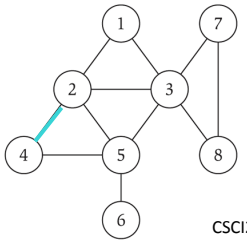
- n×n matrix with A_{uv} = 1 if (u, v) is an edge
  - Two representations of each edge (symmetric matrix)
  - Space?
  - Checking if (u, v) is an edge?
  - Identifying all edges?



	1	2	3	4	5	6	7	8
1	0	1	1	0	0	0	0	0
2	1	0	1	1	1	0	0	0
3	1	1	0	0	1	0	1	1
4	0	1	0	0	1	0	0	0
5	0	1	1	1	0	1	0	0
6	0	0	0	0	1	0	0	0
7	0	0	1	0	0	0	0	1
8	0	0	1	0	0	0	1	0

Graph Representation: Adjacency Matrix

- n×n matrix with A_{uv} = 1 if (u, v) is an edge
  - Two representations of each edge (symmetric matrix)
  - Space: $\Theta(n^2)$
  - Checking if (u, v) is an edge: $\Theta(1)$ time
  - Identifying all edges: $\Theta(n^2)$ time

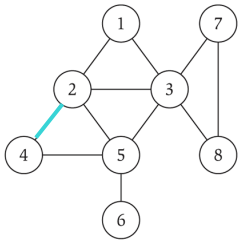


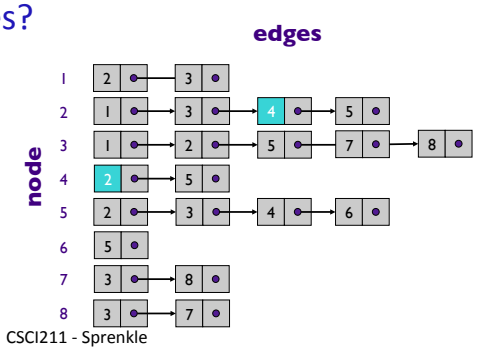
	1	2	3	4	5	6	7	8
1	0	1	1	0	0	0	0	0
2	1	0	1	1	1	0	0	0
3	1	1	0	0	1	0	1	1
4	0	1	0	0	1	0	0	0
5	0	1	1	1	0	1	0	0
6	0	0	0	0	1	0	0	0
7	0	0	1	0	0	0	0	1
8	0	0	1	0	0	0	1	0

Graph Representation: Adjacency List

- Node indexed array of lists
  - Two representations of each edge
  - Space?
  - Checking if (u, v) is an edge?
  - Identifying all edges?

What are the extremes?

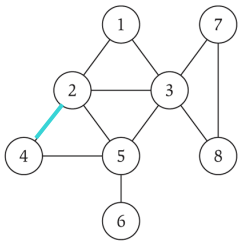


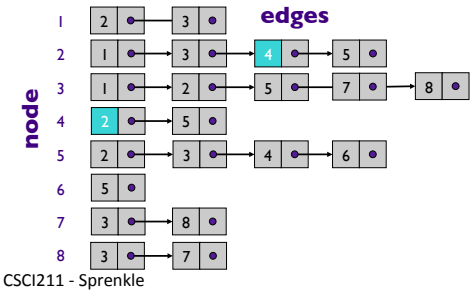


Graph Representation: Adjacency List

- Node indexed array of lists
  - Two representations of each edge
  - Space = 2m + n = O(m + n)
  - Checking if (u, v) is an edge takes O(deg(u)) time
  - Identifying all edges takes Θ(m + n) time

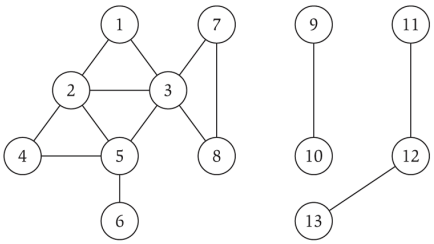
degree = number of neighbors of u





Paths and Connectivity

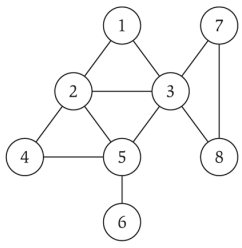
- Def. A ***path*** in an undirected graph G = (V, E) is a sequence P of nodes v_1, v_2, …, v_{k-1}, v_k
  - Each consecutive pair v_i, v_{i+1} is joined by an edge in E
- Def. A path is ***simple*** if all nodes are *distinct*
- Def. An undirected graph is ***connected*** if ∀ pair of nodes u and v, there is a path between u and v



- *Short path*
- *Distance*

Cycles

- Def. A ***cycle*** is a path v_1, v_2, …, v_{k-1}, v_k in which $v_1 = v_k$, k > 3, and the first k-1 nodes are all distinct

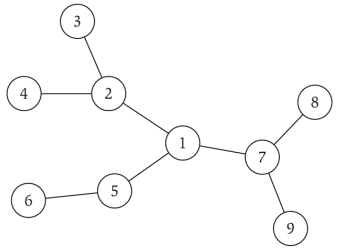


cycle C = 1-2-4-5-3-1

TREES

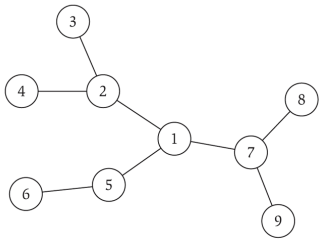
Trees

- Def. An undirected graph is a ***tree*** if it is connected and does not contain a cycle
- Simplest connected graph
  - Deleting any edge from a tree will disconnect it

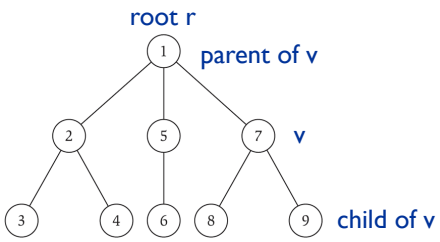


Rooted Trees

- Given a tree T, choose a root node *r* and orient each edge away from *r*
- Models hierarchical structure



a tree

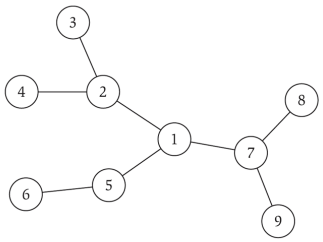


the same tree, rooted at 1

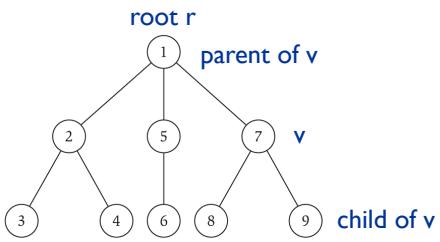
Why *n-1* edges?

Rooted Trees

- Why *n-1* edges?
  - Each non-root node has an edge to its parent



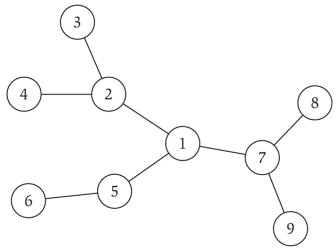
a tree



the same tree, rooted at 1

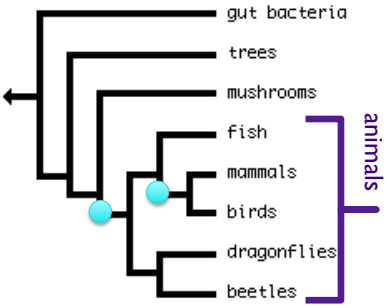
Trees

- Theorem. Let G be an undirected graph on *n* nodes. Any two of the following statements imply the third:
  - G is connected
  - G does not contain a cycle
  - G has *n*-1 edges



Phylogeny Trees

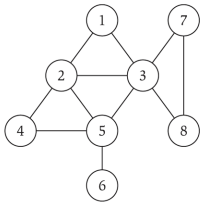
- Describe evolutionary history of species
  - mammals and birds share a common ancestor that they do not share with other species
  - all animals are descended from an ancestor not shared with mushrooms, trees, and bacteria



GRAPH CONNECTIVITY & TRAVERSAL

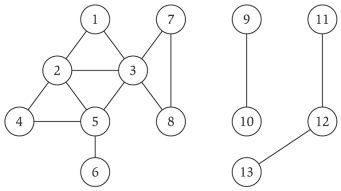
Connectivity

- **s-t connectivity problem.** Given nodes *s* and *t*, is there a path between *s* and *t*?
- **s-t shortest path problem.** Given nodes *s* and *t*, what is the length of the shortest path between *s* and *t*?
- Applications
  - Facebook
  - Maze traversal
  - Kevin Bacon number
  - Spidering the web
  - Fewest number of hops in a communication network



Application: Connected Component

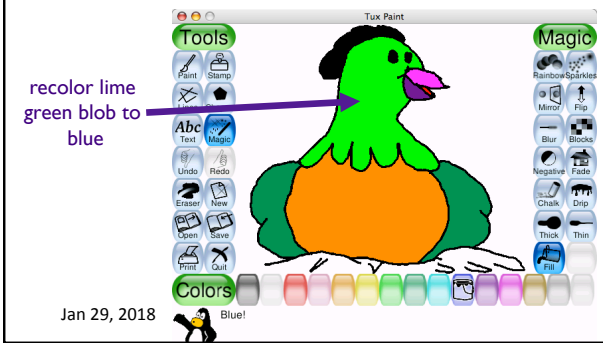
- Find all nodes ***reachable*** from *s*

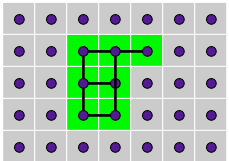


- Connected component containing node 1 is { 1, 2, 3, 4, 5, 6, 7, 8 }

Application: Flood Fill

- Given lime green pixel in an image, change color of entire blob of neighboring lime pixels to blue
  - Node: pixel
  - Edge: two neighboring lime pixels
  - Blob: connected component of lime pixels





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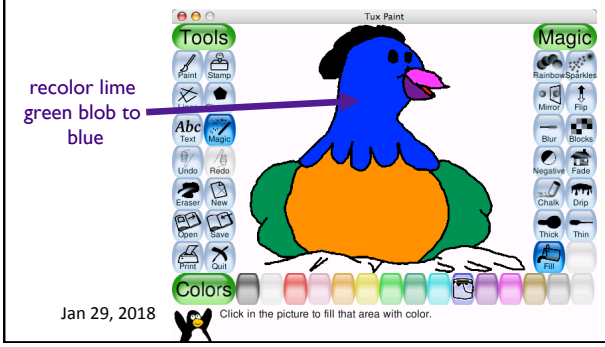
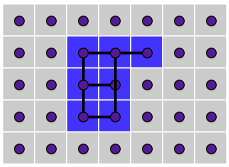
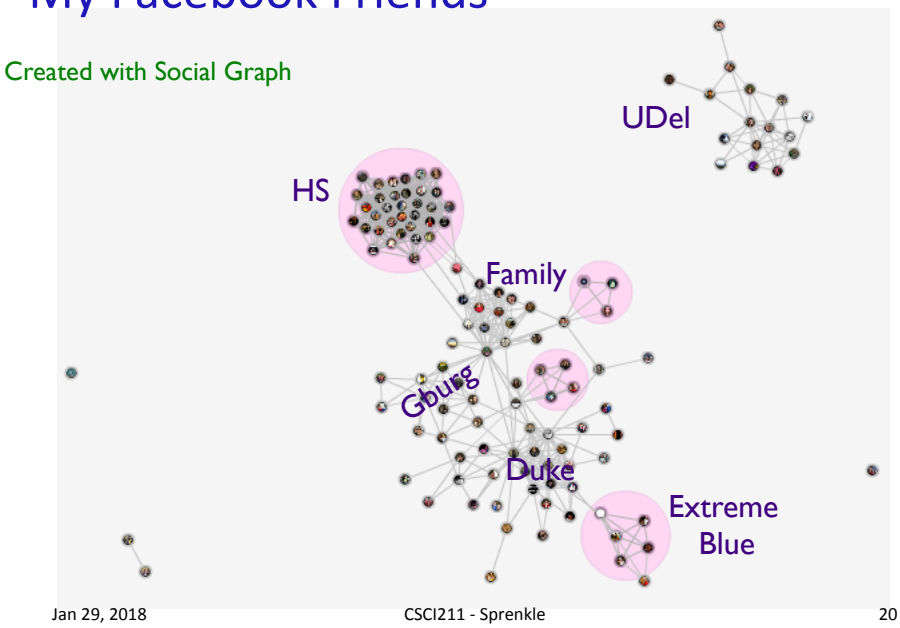


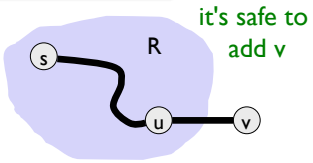


My Facebook Friends



A General Algorithm

```
R will consist of nodes to which s has a path
R = {s}
while there is an edge (u,v) where u∈R and v∉R
      add v to R
```



- *R* will be the **connected component** containing s
- Algorithm is underspecified

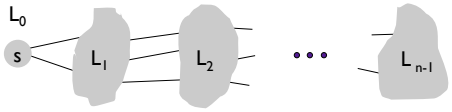
In what order should we consider the edges?

Possible Orders

- Breadth-first
- Depth-first

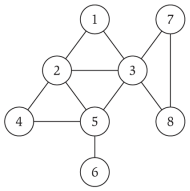
Breadth-First Search

- **Intuition**. Explore outward from *s* in all possible directions (edges), adding nodes one "layer" at a time
- **Algorithm**
  - $L_0 = \{ s \}$
  - L_1 = all neighbors of L_0
  - L_2 = all nodes that have an edge to a node in L_1 and do not belong to L_0 or L_1
  - L_{i+1} = all nodes that have an edge to a node in L_i and do not belong to an earlier layer



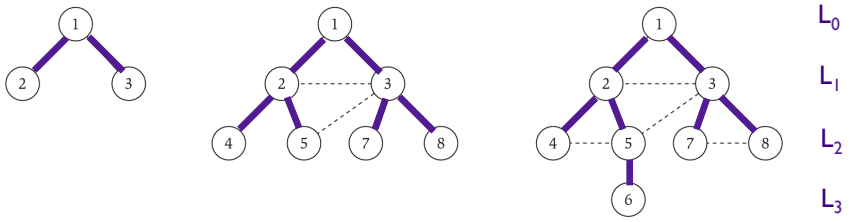
Run BFS on This Graph

s = 1



Example of Breadth-First Search

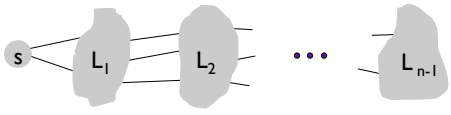
s = 1



Creates a tree
-- is a node in the graph that is not in the tree

Breadth-First Search

- **Theorem.**
  For each *i*, L_i consists of all nodes at distance exactly *i* from *s*.
  There is a path from s to t iff t appears in some layer.



- What does this theorem mean?
- Can we determine the distance between s and t?

Looking Ahead

- Monday, 11:59 p.m.: journal - Chapter 2.4, 2.5, 3.1
- Friday: Problem Set 3 due