Objectives

- Wrap up: Implementing BFS and DFS
- Graph Application: Bipartite Graphs

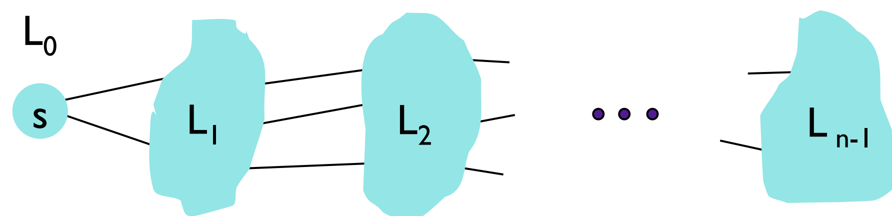
Get out your BFS implementation handouts

Review

- What are two ways to find a connected component?
  - How are their results similar? Different?

Review: Breadth-First Search

- **Intuition.** Explore outward from *s* in all possible directions (edges), adding nodes one "layer" at a time



- **Algorithm**
  - L_0 = { s }
  - L_1 = all neighbors of L_0
  - L_2 = all nodes that have an edge to a node in L_1 and do not belong to L_0 or L_1
  - L_{i+1} = all nodes that have an edge to a node in L_i and do not belong to an earlier layer

Analysis

n

$O(n^3)$

At most n

At most n-1

At most n-1

```
BFS(s, G):
    Discovered[v] = false, for all v
    Discovered[s] = true
    L[0] = {s}
    layer counter i = 0
    BFS tree T = {}
    while L[i] != {}
        L[i+1] = {}
        For each node u ∈ L[i]
            Consider each edge (u,v) incident to u
            if Discovered[v] == false then
                Discovered[v] = true
                Add edge (u, v) to tree T
                Add v to the list L[i + 1]
        i+=1
```

Analysis: Tighter Bound

n

$O(n^2)$

At most n

At most n-1

```
BFS(s, G):
    Discovered[v] = false, for all v
    Discovered[s] = true
    L[0] = {s}
    layer counter i = 0
    BFS tree T = {}
    while L[i] != {}
        L[i+1] = {}
        For each node u ∈ L[i]
            Consider each edge (u,v) incident to u
            if Discovered[v] == false then
                Discovered[v] = true
                Add edge (u, v) to tree T
                Add v to the list L[i + 1]
        i+=1
```

Because we're going to look at each node at most once

Analysis: Even Tighter Bound

```
BFS(s, G):
    Discovered[v] = false, for all v
    Discovered[s] = true
    L[0] = {s}
    layer counter i = 0
    BFS tree T = {}
    while L[i] != {}
        L[i+1] = {}
        For each node u ∈ L[i]
            Consider each edge (u,v) incident to u
            if Discovered[v] == false then
                Discovered[v] = true
                Add edge (u, v) to tree T
                Add v to the list L[i + 1]
        i+=1
```

n

At most n

O(deg(u))

$\Sigma_{u \in V} \deg(u) = 2m$

→ O(n+m)

Implementing DFS

- What do we need as input?
- What do we need to model?
  - How will we model that?
  - Pseudo code

```
DFS(u):
   Mark u as “Explored” and add u to R
   For each edge (u, v) incident to u
      If v is not marked “Explored” then
         DFS(v)
```

Implementing DFS

- Keep nodes to be processed in a *stack*

```
DFS(s, G):
    Initialize S to be a stack with one element s
    Explored[v] = false, for all v
    Parent[v] = 0, for all v
    DFS tree T = {}
    while S != {}
        Take a node u from S
        if Explored[u] = false
            Explored[u] = true
            Add edge (u, Parent[u]) to T (if u ≠ s)
            for each edge (u, v) incident to u
                Add v to the stack S
                Parent[v] = u
```

What is the runtime?
How many times is a node added/removed from the stack?

Analyzing DFS

O(n+m)

```
DFS(s, G):
    Initialize S to be a stack with one element s
    Explored[v] = false, for all v  } O(n)
    Parent[v] = 0, for all v
    DFS tree T = {}
    while S != {}
        Take a node u from S
        if Explored[u] = false
            Explored[u] = true
            Add edge (u, Parent[u]) to T (if u ≠ s)
   deg(u)   for each edge (u, v) incident to u
                Add v to the stack S
                Parent[v] = u
```

A node is added/removed from the stack 2* deg(u)

All nodes are added 2m = O(m) times

Analyzing Finding All Connected Components

- How can we find the set of all connected components of the graph?

```
R* = set of connected components (a set of sets)

while there is a node that does not belong to R*

    select s not in R*

    R = {s}

    while there is an edge (u,v) where u∈R and v∉R
        add v to R


    Add R to R*
```

But the inner loop is O(m+n)!
How can this RT be possible?

Claim: Running time is O(m+n)

Set of All Connected Components

- How can we find the set of all connected components of the graph?

```
R* = set of connected components (a set of sets)

while there is a node that does not belong to R*

    select s not in R*

    R = {s}

    while there is an edge (u,v) where u∈R and v∉R
        add v to R


    Add R to R*
```

Imprecision in the running time of inner loop: O(m+n)

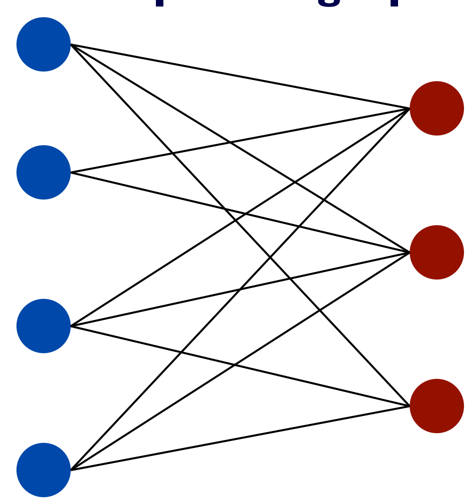
But that's m and n of the ***connected*** component, let's say m_i and n_i .
Σ_i O(m_i+ n_i) = O(m+n)

Where *i* is the subscript of the connected component

BIPARTITE GRAPHS

Bipartite Graphs

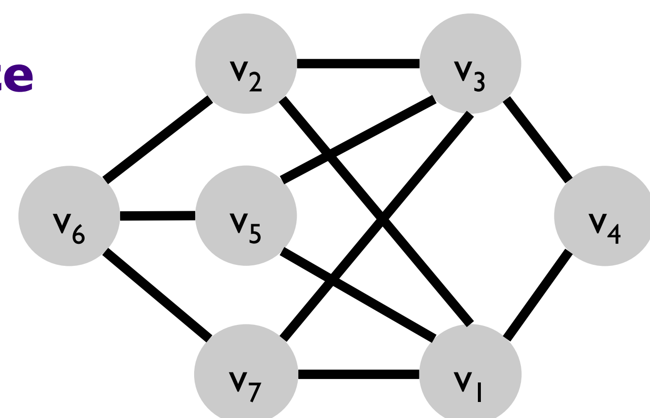
- Def. An undirected graph G = (V, E) is ***bipartite*** if the nodes can be colored red or blue such that every edge has one red and one blue end
  - Generally: vertices divided into sets X and Y
- Applications:
  - Stable marriage:
    - men = red, women = blue
  - Scheduling:
    - machines = red, jobs = blue



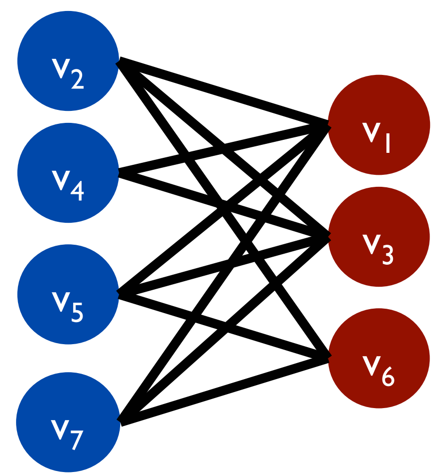
Testing Bipartiteness

- Given a graph G, is it bipartite?
- Many graph problems become:
  - Easier if underlying graph is bipartite (e.g., matching)
  - Tractable if underlying graph is bipartite (e.g., independent set)
- Before designing an algorithm, need to understand structure of bipartite graphs

a bipartite graph G:



another drawing of G:



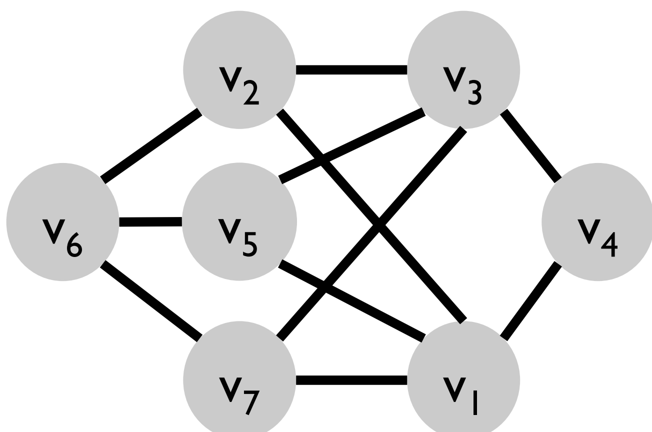
How Can We Determine if a Graph is Bipartite?

- Given a connected graph

Why connected?

  1. Color one node red
      - Doesn't matter which color (Why?)

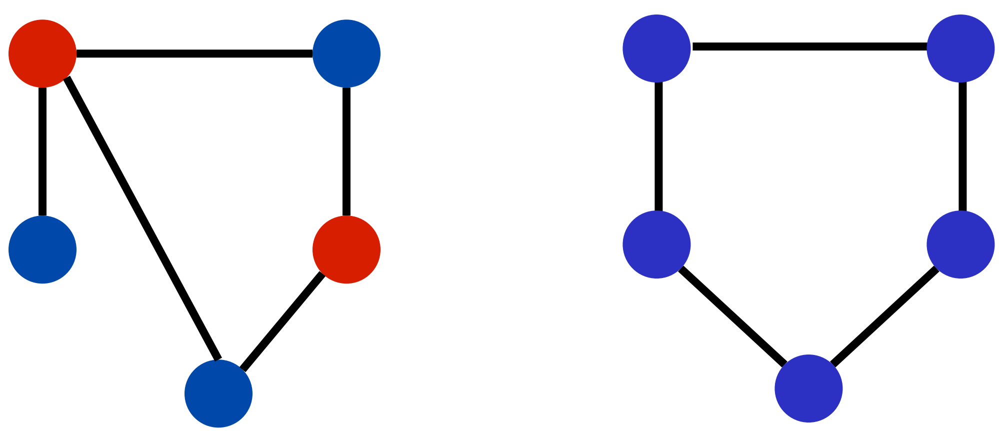
  ➢ What should we do next?



- How will we know when we're finished?
- What does this process sound like?

An Obstruction to Bipartiteness

- Lemma. If a graph G is bipartite, it cannot contain an odd-length cycle.

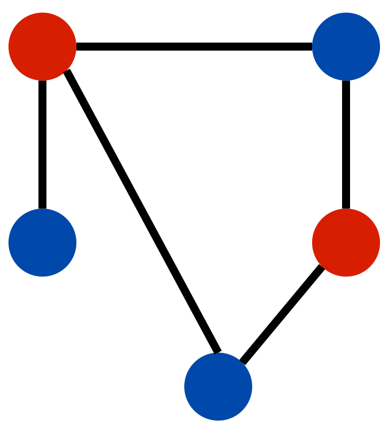


bipartite (2-colorable)

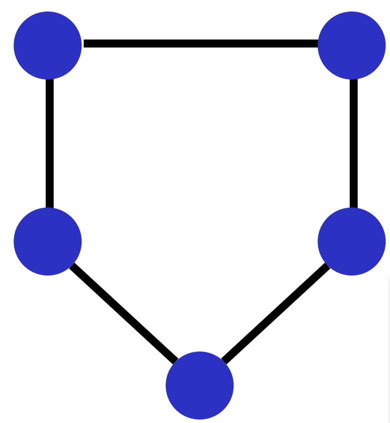
not bipartite (not 2-colorable)

An Obstruction to Bipartiteness

- Lemma. If a graph G is bipartite, it cannot contain an odd-length cycle.
- Pf. Not possible to 2-color the odd cycle, let alone G.



bipartite
(2-colorable)

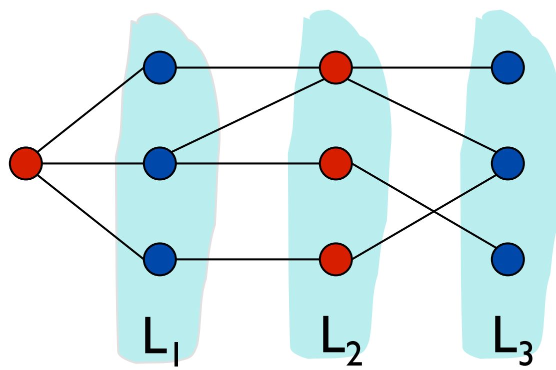


not bipartite
(not 2-colorable)

If find an odd cycle, graph is NOT bipartite

How Can We Determine if a Graph is Bipartite?

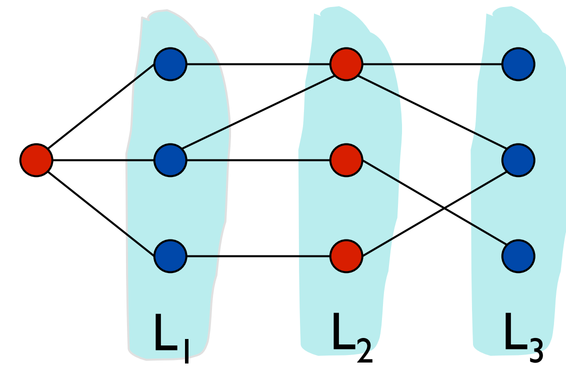
- Given a connected graph
  - Color one node red
    - Doesn't matter which color (Why?)
  - What should we do next?
- How will we know that we're finished?
- What does this process sound like?
  - BFS: alternating colors, layers



How can we implement the algorithm?

Implementing Algorithm

- Modify BFS to have a Color array
- When add v to list L[i+1]
  - Color[v] = red if i+1 is even
  - Color[v] = blue if i+1 is odd



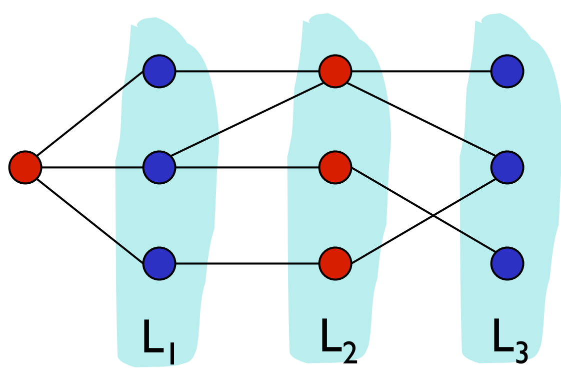
What is the running time of this algorithm? **O(n+m)**

Marks a change in how we think about algorithms
Starting to apply known algorithms to solve new problems

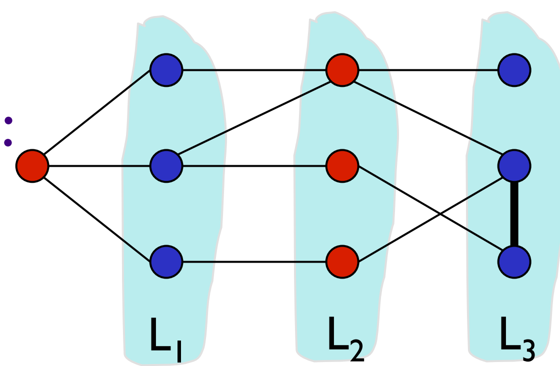
Analyzing Algorithm's Correctness

- Lemma. Let G be a connected graph, and let L_0, …, L_k be the layers produced by BFS starting at node *s*. Exactly one of the following holds:
  - (i) No edge of G joins two nodes of the same layer
    - ⇨ G is bipartite
  - (ii) An edge of G joins two nodes of the same layer
    - ⇨ G contains an odd-length cycle and hence is not bipartite

Case (i):



Case (ii):



Analyzing Algorithm's Correctness

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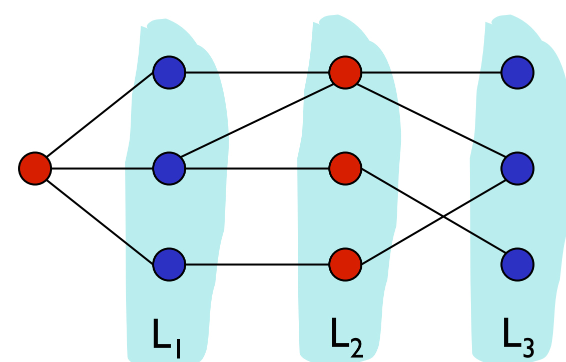
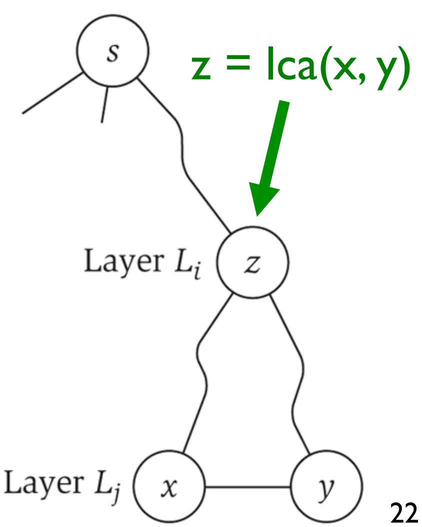
  - (i) No edge of G joins two nodes of the same layer

  ➡ G is bipartite
- Pf. (i)
  - Suppose no edge joins two nodes in the same layer
  - Implies all edges join nodes on adjacent level
  - Bipartition
    - red = nodes on odd levels
    - blue = nodes on even levels



Analyzing Algorithm's Correctness

- Lemma. Let G be a connected graph, and let L_0, …, L_k be the layers produced by BFS starting at node *s*. Exactly one of the following holds:
  - (ii) An edge of G joins two nodes of the same layer → G contains an odd-length cycle and hence is not bipartite
- Pf. (ii)
  - Suppose (x, y) is an edge with x, y in same level L_j.
  - Let z = lca(x, y) = lowest common ancestor
  - Let L_i be level containing z
  - Consider cycle that takes edge from x to y, then path y →z, then path from z → x



Analyzing Algorithm's Correctness

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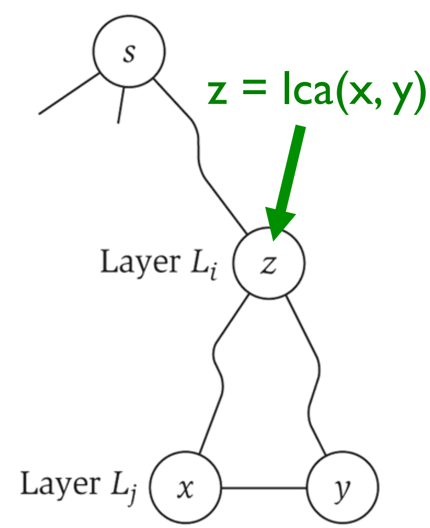
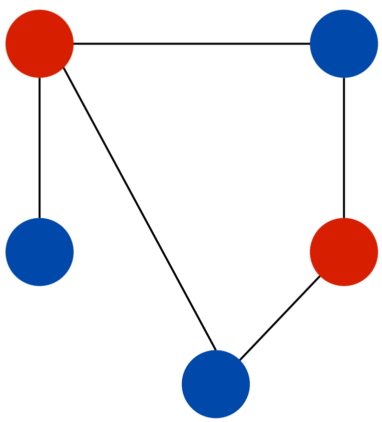
  - Let z = lca(x, y)=lowest common ancestor
  - Let L_i be level containing z
  - Consider cycle that takes edge from x to y, then path y → z, then path z → x
  - Its length is $\underbrace{1}_{(x,\,y)} + \underbrace{(j-i)}_{\text{path from } y \text{ to } z} + \underbrace{(j-i)}_{\text{path from } z \text{ to } x}$, which is odd

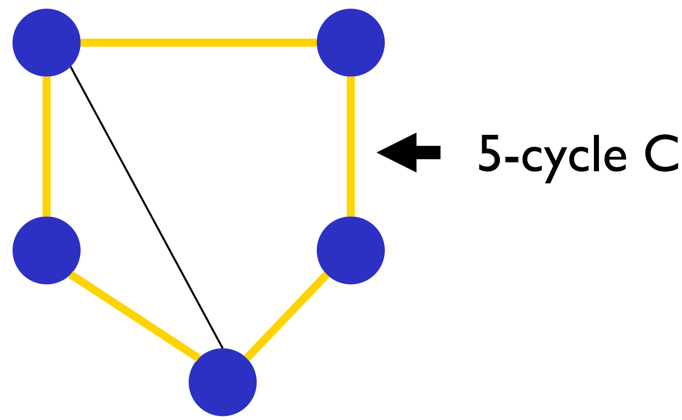


An Obstruction to Bipartiteness

- Corollary. A graph G is bipartite *iff* it contains no odd length cycle.



bipartite
(2-colorable)



not bipartite
(not 2-colorable)

Looking Ahead

Goal: Finish graphs before Exam 1

- Wiki: 3.2-3.6
  - Covered in class: 3.2-3.4
  - Expected: 3.5-3.6 on Monday
  - Willing to push wiki to Tuesday at 11:59 p.m.
- PS 4 – due Friday
  - First two problems – know how to do now
  - Second two problems should wait until after Monday’s class