# Objectives

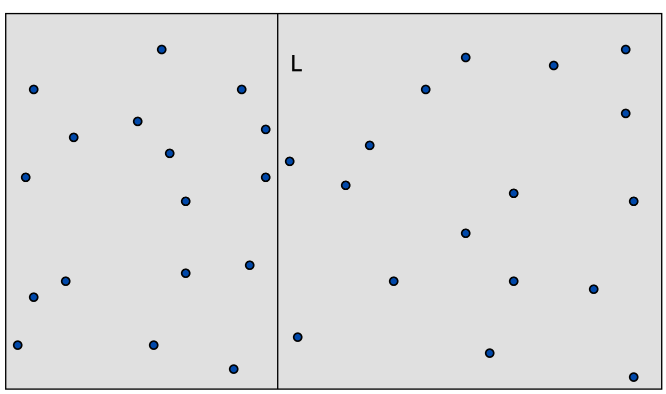
- Divide and conquer
  - Closest pair of points
  - Integer multiplication
  - Matrix multiplication



# Review: Closest Pair of Points

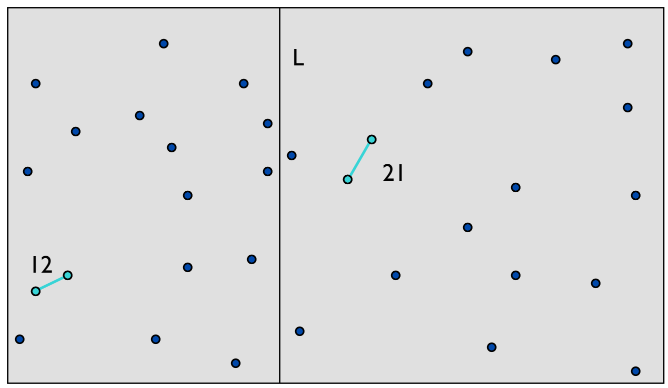
- Divide: draw vertical line L so that roughly ½n points on each side

How do we implement this?



## Review: Closest Pair of Points

- Divide: draw vertical line L so that roughly ½n points on each side
- Conquer: find closest pair in each side recursively



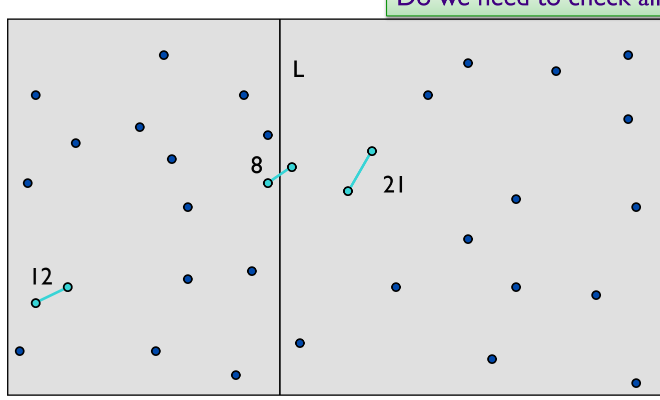



## Closest Pair of Points

- Divide: draw vertical line L so that roughly ½n points on each side
- Conquer: find closest pair in each side recursively
- Combine: find closest pair with one point in each side
- Return best of 3 solutions

*seems like $\Theta(n^2)$*

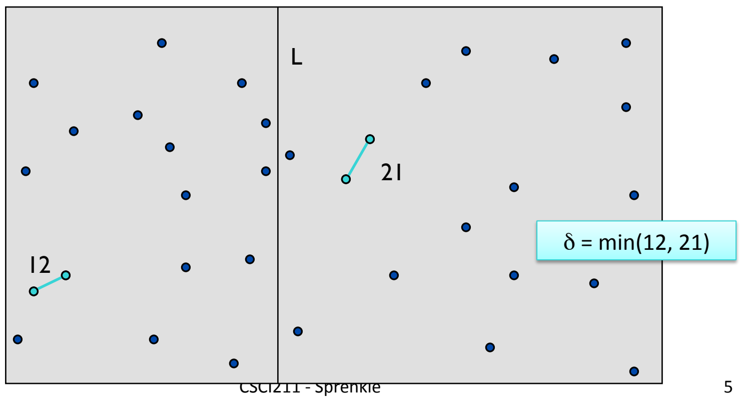
Do we need to check all pairs?



## Closest Pair of Points

- Find closest pair with one point in each side, assuming that distance < δ

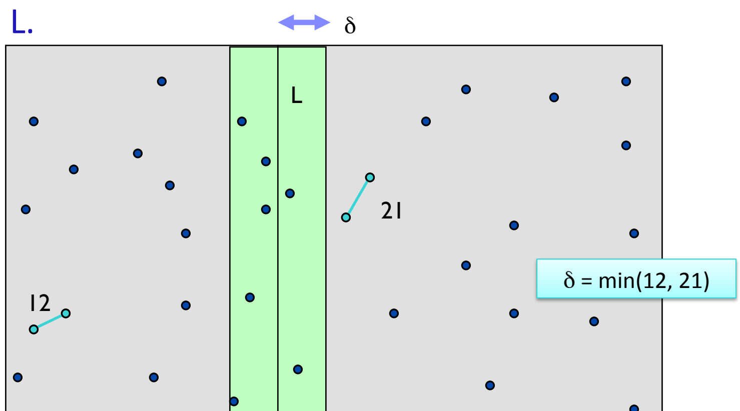
  where δ = min(left_min_dist, right_min_dist)





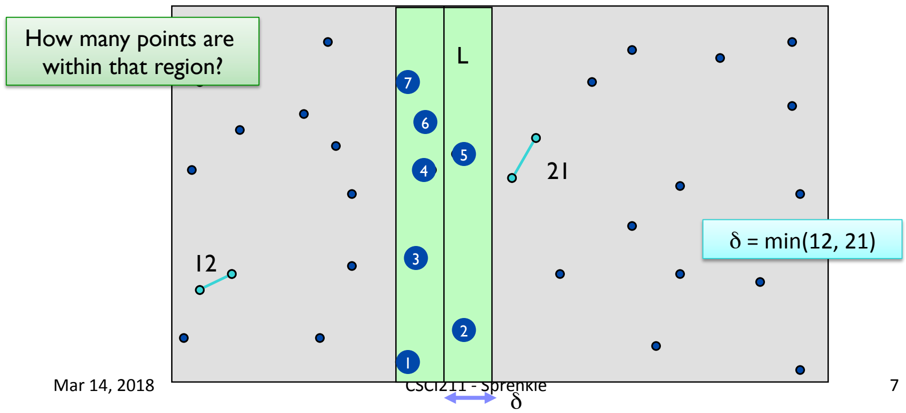

## Closest Pair of Points

- Find closest pair with one point in each side, assuming that distance < δ.
  - Observation: only need to consider points within δ of line L.



# Closest Pair of Points

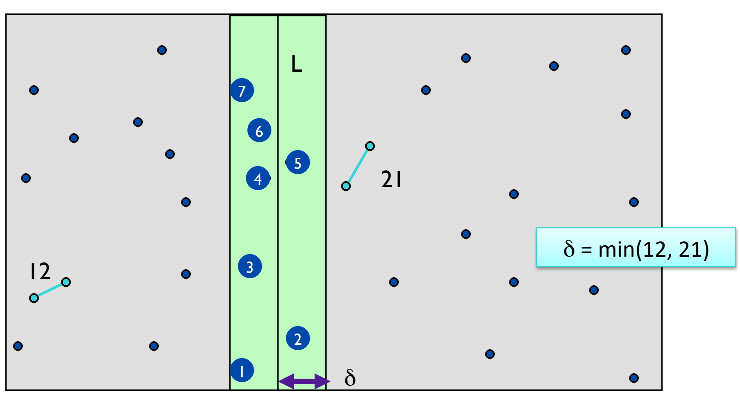
- Find closest pair w/ 1 point in each side, assuming that distance < δ.
  - Observation: only consider points within δ of line L
  - Sort points in 2δ-strip by their y coordinate






# Closest Pair of Points

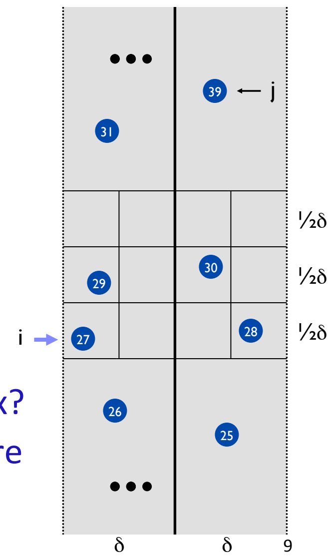
- Find closest pair w/ 1 point in each side, assuming that distance < δ
  - Observation: only consider points within δ of line L
  - Sort points in 2δ-strip by their y coordinate
    - Only checks distances of those within 11 positions in sorted list!



# Analyzing Cost of Combining

Prepare minds to be blown…

- Def. Let $s_i$ be the point in the $2\delta$-strip, with the $i^{th}$ smallest y-coordinate
- Claim. If $|i - j| \geq 12$, then the distance between $s_i$ and $s_j$ is at least $\delta$
  - What is the distance of the box?
  - How many points can be in a box?
  - When do we know that points are $> \delta$ apart?

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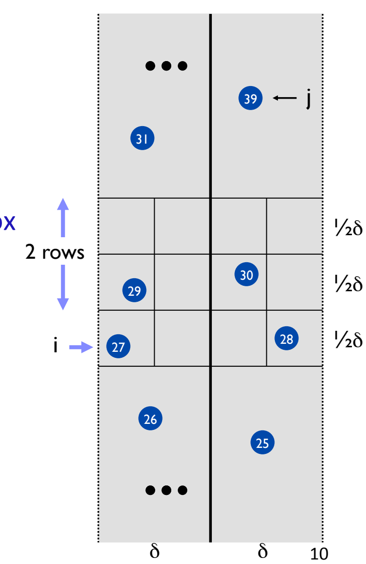
# Analyzing Cost of Combining

- Def. Let $s_i$ be the point in the $2\delta$-strip, with the $i^{th}$ smallest y-coordinate
- Claim. If $|i - j| \geq 12$, then the distance between $s_i$ and $s_j$ is at least $\delta$
- Pf.
  - No two points lie in same ½δ-by-½δ box
  - Two points at least 2 rows apart have distance ≥ 2(½δ). ▪
- Fact. Still true if we replace 12 with 7.




Cost of combining is therefore…?

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# Closest Pair Algorithm

```
Closest-Pair(p1, …, pn)
   Compute separation line L such that half the points
   are on one side and half on the other side.

   δ1 = Closest-Pair(left half)
   δ2 = Closest-Pair(right half)
   δ  = min(δ1, δ2)

   Delete all points further than δ from separation
     line L

   Sort remaining points by y-coordinate.

   Scan points in y-order and compare distance between
   each point and next 7 neighbors. If any of these
   distances is less than δ, update δ.

   return δ
```



# Closest Pair Algorithm

```
Closest-Pair(p1, …, pn)
   Compute separation line L such that half the points     O(n log n)
   are on one side and half on the other side.

   δ1 = Closest-Pair(left half)                            2T(n / 2)
   δ2 = Closest-Pair(right half)
   δ  = min(δ1, δ2)

   Delete all points further than δ from separation        O(n)
     line L

   Sort remaining points by y-coordinate.                  O(n log n)

   Scan points in y-order and compare distance between
   each point and next 7 neighbors. If any of these        O(n)
   distances is less than δ, update δ.

   return δ
```

Putting the recurrence relation together…

$T(n) = 2T(n/2) + O(n \log n)$

## Closest Pair of Points: Analysis

- Running time. Solved in 5.2

$$T(n) \le 2T(n/2) + O(n \log n) \Rightarrow T(n) = O(n \log^2 n)$$

- Can we achieve O(n log n)?

$$T(n) \le 2T(n/2) + O(n) \Rightarrow T(n) = O(n \log n)$$

- Yes. Don't sort points in strip from scratch each time.
  - Each recursive call returns two lists: all points sorted by y coordinate, and all points sorted by x coordinate
  - Sort by merging two pre-sorted lists



## INTEGER AND MATRIX MULTIPLICATION

# Integer Arithmetic

- Add. Given 2 *n*-digit integers a and b, compute a + b.
  - Algorithm?
  - Runtime?

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 | 1 | 0 | 1 | | |
| | 1 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | a |
| + | 0 | 1 | 1 | 1 | 1 | 1 | 0 | 1 | b |
| 1 | 0 | 1 | 0 | 1 | 0 | 0 | 1 | 0 | a + b |



# Integer Arithmetic

- Add. Given 2 *n*-digit integers a and b, compute a + b.
  - Algorithm?
  - Runtime?

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 | 1 | 0 | 1 | | |
| | 1 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | a |
| + | 0 | 1 | 1 | 1 | 1 | 1 | 0 | 1 | b |
| 1 | 0 | 1 | 0 | 1 | 0 | 0 | 1 | 0 | a + b |

O(n) operations

# Integer Arithmetic

- Multiply. Given 2 *n*-digit integers a and b, compute a × b.
  - Algorithm?
  - Runtime?

```
   1 1 0 1 0 1 0 1    a
 * 0 1 1 1 1 1 0 1    b
                     a × b
```



# Integer Arithmetic

- Multiply. Given 2 *n*-digit integers a and b, compute a × b.
  - Brute force solution: $\Theta(n^2)$ bit operations

Goal: Faster algorithm

```
                1 1 0 1 0 1 0 1
              * 0 1 1 1 1 1 0 1
              -------------------
                1 1 0 1 0 1 0 1
              0 0 0 0 0 0 0 0
            1 1 0 1 0 1 0 1
          1 1 0 1 0 1 0 1
        1 1 0 1 0 1 0 1
      1 1 0 1 0 1 0 1
    1 1 0 1 0 1 0 1
  0 0 0 0 0 0 0 0
  -------------------------------
  1 1 0 1 0 0 0 0 0 0 0 0 0 0 0 1
```

## Divide-and-Conquer Multiplication: Warmup

- To multiply 2 n-digit integers:
  - Multiply 4 ½n-digit integers
  - Add 2 ½n-digit integers and shift to obtain result

Higher order bits — Lower order bits — Shift

x=10001101
$x_1$=1000 $x_0$=1101

$$x = 2^{n/2}\cdot x_1 + x_0$$
$$y = 2^{n/2}\cdot y_1 + y_0$$
$$xy = \left(2^{n/2}\cdot x_1 + x_0\right)\left(2^{n/2}\cdot y_1 + y_0\right) = \underbrace{2^n\cdot x_1y_1}_{A} + 2^{n/2}\cdot(\underbrace{x_1y_0}_{B} + \underbrace{x_0y_1}_{C}) + \underbrace{x_0y_0}_{D}$$

What is the recurrence relation?
- How many subproblems?
- What is merge cost?
- What is its runtime?



## Divide-and-Conquer Multiplication: Warmup

- To multiply 2 n-digit integers:
  - Multiply 4 ½n-digit integers
  - Add 2 ½n-digit integers and shift to obtain result

Higher order bits — Lower order bits — Shift

$$x = 2^{n/2}\cdot x_1 + x_0$$
$$y = 2^{n/2}\cdot y_1 + y_0$$
$$xy = \left(2^{n/2}\cdot x_1 + x_0\right)\left(2^{n/2}\cdot y_1 + y_0\right) = \underbrace{2^n\cdot x_1y_1}_{A} + 2^{n/2}\cdot(\underbrace{x_1y_0}_{B} + \underbrace{x_0y_1}_{C}) + \underbrace{x_0y_0}_{D}$$

$$T(n) = \underbrace{4T(n/2)}_{\text{recursive calls}} + \underbrace{\Theta(n)}_{\text{add, shift}} \Rightarrow T(n) = \Theta(n^2)$$

assumes n is a power of 2

Not an improvement over brute force

# Karatsuba Multiplication

Anatolii Alexeevich Karatsuba

- To multiply two n-digit integers:
  - Add 2 ½n digit integers
  - Multiply 3 ½n-digit integers
  - Add, subtract, and shift ½n-digit integers to obtain result

$$
\begin{aligned}
x &= 2^{n/2} \cdot x_1 + x_0 \\
y &= 2^{n/2} \cdot y_1 + y_0 \\
xy &= 2^{n} \cdot x_1 y_1 + 2^{n/2} \cdot (x_1 y_0 + x_0 y_1) + x_0 y_0 \\
&= 2^{n} \cdot \underbrace{x_1 y_1}_{A} + 2^{n/2} \cdot \left( \underbrace{(x_1 + x_0)(y_1 + y_0)}_{B} - \underbrace{x_1 y_1}_{A} - \underbrace{x_0 y_0}_{C} \right) + \underbrace{x_0 y_0}_{C}
\end{aligned}
$$

What is the recurrence relation? Runtime?



# Karatsuba Multiplication

- Theorem. [Karatsuba-Ofman, 1962] Can multiply two n-digit integers in $O(n^{1.585})$ bit operations

$$
\begin{aligned}
x &= 2^{n/2} \cdot x_1 + x_0 \\
y &= 2^{n/2} \cdot y_1 + y_0 \\
xy &= 2^{n} \cdot x_1 y_1 + 2^{n/2} \cdot (x_1 y_0 + x_0 y_1) + x_0 y_0 \\
&= 2^{n} \cdot \underbrace{x_1 y_1}_{A} + 2^{n/2} \cdot \left( \underbrace{(x_1 + x_0)(y_1 + y_0)}_{B} - \underbrace{x_1 y_1}_{A} - \underbrace{x_0 y_0}_{C} \right) + \underbrace{x_0 y_0}_{C}
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{T}(n) &\le \underbrace{T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + T(1 + \lceil n/2 \rceil)}_{\text{recursive calls}} + \underbrace{\Theta(n)}_{\text{add, subtract, shift}} \\
\Rightarrow \mathrm{T}(n) &= O(n^{\log_2 3}) = O(n^{1.585})
\end{aligned}
$$

# MATRIX MULTIPLICATION



## Matrix Multiplication

- Given 2 n-by-n matrices A and B, compute C = AB

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

$$\begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \times \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix}$$

  - Ex: $c_{12} = a_{11} b_{12} + a_{12} b_{22} + a_{13} b_{32} + \ldots + a_{1n} b_{n2}$

Row 1 of a Column 2 of b

Solve using brute force …

## Matrix Multiplication

- Given 2 n-by-n matrices A and B, compute C = AB

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

$$\begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \times \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix}$$

  - Ex: $c_{12} = a_{11} b_{12} + a_{12} b_{22} + a_{13} b_{32} + \ldots + a_{1n} b_{n2}$

- Brute force. $\Theta(n^3)$ arithmetic operations
- Fundamental question: Can we improve upon brute force?



## Matrix Multiplication: Warmup

- Divide: partition A and B into ½n-by-½n blocks
- Conquer: multiply 8 ½n-by-½n recursively
- Combine: add appropriate products using 4 matrix additions

$$\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \times \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

$$\begin{aligned} C_{11} &= (A_{11} \times B_{11}) + (A_{12} \times B_{21}) \\ C_{12} &= (A_{11} \times B_{12}) + (A_{12} \times B_{22}) \\ C_{21} &= (A_{21} \times B_{11}) + (A_{22} \times B_{21}) \\ C_{22} &= (A_{21} \times B_{12}) + (A_{22} \times B_{22}) \end{aligned}$$

Recurrence relation? Runtime?

# Matrix Multiplication: Warmup

- Divide: partition A and B into ½n-by-½n blocks
- Conquer: multiply 8 ½n-by-½n recursively
- Combine: add appropriate products using 4 matrix additions

$$\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \times \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

$$\begin{aligned} C_{11} &= (A_{11} \times B_{11}) + (A_{12} \times B_{21}) \\ C_{12} &= (A_{11} \times B_{12}) + (A_{12} \times B_{22}) \\ C_{21} &= (A_{21} \times B_{11}) + (A_{22} \times B_{21}) \\ C_{22} &= (A_{21} \times B_{12}) + (A_{22} \times B_{22}) \end{aligned}$$

$$\mathrm{T}(n) = \underbrace{8T(n/2)}_{\text{recursive calls}} + \underbrace{\Theta(n^2)}_{\text{add, form submatrices}} \Rightarrow \mathrm{T}(n) = \Theta(n^3)$$



# Matrix Multiplication: Key Idea

Trade expensive multiplication for less expensive addition/subtraction

- Multiply 2-by-2 block matrices with only 7 multiplications and 15 additions

$$\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \times \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

$$\begin{aligned} C_{11} &= P_5 + P_4 - P_2 + P_6 \\ C_{12} &= P_1 + P_2 \\ C_{21} &= P_3 + P_4 \\ C_{22} &= P_5 + P_1 - P_3 - P_7 \end{aligned}$$

$$\begin{aligned} P_1 &= A_{11} \times (B_{12} - B_{22}) \\ P_2 &= (A_{11} + A_{12}) \times B_{22} \\ P_3 &= (A_{21} + A_{22}) \times B_{11} \\ P_4 &= A_{22} \times (B_{21} - B_{11}) \\ P_5 &= (A_{11} + A_{22}) \times (B_{11} + B_{22}) \\ P_6 &= (A_{12} - A_{22}) \times (B_{21} + B_{22}) \\ P_7 &= (A_{11} - A_{21}) \times (B_{11} + B_{12}) \end{aligned}$$

## Fast Matrix Multiplication
[Strassen, 1969]

Volker Strassen

- Divide: partition A and B into ½n-by-½n blocks
- Compute: 14 ½n-by-½n matrices via 10 matrix additions
- Conquer: multiply 7 ½n-by-½n matrices recursively
- Combine: 7 products into 4 terms using 8 matrix additions
- Analysis.
  - Assume n is a power of 2.
  - T(n) = # arithmetic operations.

$$T(n) = \underbrace{7T(n/2)}_{\text{recursive calls}} + \underbrace{\Theta(n^2)}_{\text{add, subtract}} \Rightarrow T(n) = \Theta(n^{\log_2 7}) = O(n^{2.81})$$

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## Another Approach to Solving Recurrences: Master Method

- General approach to solving recurrences
  - Not covered in our text book
- Given a recurrence T(n) = a T (n/b) + $n^c$, the running time is
  - (nlogba) when a > bc
  - (Nclogba) when a = bc
  - (nc) when a < bc

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## Fast Matrix Multiplication in Practice

- Implementation issues:
  problems putting theory into practice
  - Sparsity
  - Caching effects
  - Numerical stability
    - Theoretically correct but possible problems with round off errors, etc
  - Odd matrix dimensions
- Crossover to classical algorithm around
  n = 128



## Fast Matrix Multiplication in Practice

- Common misperception:
  "Strassen is only a theoretical curiosity."
  - Advanced Computation Group at Apple Computer reports **8x** speedup on G4 Velocity Engine when n ~2,500
  - Range of instances where it's useful is a subject of controversy
- Can "Strassenize" Ax=b, determinant, eigenvalues, and other matrix ops

## Fast Matrix Multiplication in Theory

- Q. Multiply two 2-by-2 matrices with only 7 scalar multiplications?
- A. Yes! [Strassen, 1969] $\Theta(n^{\log_2 7}) = O(n^{2.81})$
- Q. Multiply two 2-by-2 matrices with only 6 scalar multiplications?
- A. Impossible [Hopcroft and Kerr, 1971] $\Theta(n^{\log_2 6}) = O(n^{2.59})$
- Q. Two 3-by-3 matrices with only 21 scalar multiplications?
- A. Also impossible $\Theta(n^{\log_3 21}) = O(n^{2.77})$
- Q. Two 70-by-70 matrices with only 143,640 scalar multiplications?
- A. Yes! [Pan, 1980] $\Theta(n^{\log_{70} 143640}) = O(n^{2.80})$

- Decimal wars.
  - December, 1979: $O(n^{2.521813})$
  - January, 1980: $O(n^{2.521801})$



## Fast Matrix Multiplication in Theory

- Best known. $O(n^{2.376})$
  [Coppersmith-Winograd, 1987]
  - But *really* large constant
- Conjecture. $O(n^{2+\varepsilon})$ for any $\varepsilon > 0$.

- Caveat. Theoretical improvements to Strassen are progressively less practical.

## Looking Ahead

- PS7 due Friday
- Exam 2 handed out Friday
- Moving to Dynamic Programming on Friday!